Content

Preface

The sixth international ProMath Conference was organized at the University of Debrecen (Hungary), 8-11 September, 2005. There were altogether 13 participants from different countries (Germany, Finland, Hungary, Slovakia) of whom most had a presentation. This volume contains a short historical introduction and almost all of the papers given on the conference.

At the meeting there were 12 presentations. The presentations had a time slot of 30 minutes with a following-up discussion of another 30 minutes. In the Proceedings the presentations are ordered alphabetically.

Problem solving has been one of the main directions in the international discussion on the teaching of mathematics. The topics of the presentations are from different areas of problem solving and mathematics instruction. We can find concrete suggestions for mathematics instruction of talented students, ideas for teacher training and some theoretical questions.

The papers in the proceedings are peer-reviewed as organized by the editor. Every paper was read and commented by two anonymous peers. The action improved remarkably the quality of most papers. However the papers are neither proof-read by the editor, nor has their language been checked. Every author is responsible for his/her own text. The e-mail addresses and the workplaces of the authors can be found in each paper.

The meetings of the research group ProMath (Problem Solving in Mathematics Education) are based on an initiative of Professor Günter Graumann (Germany), Professor Erkki Pehkonen (Finland) and Professor Bernd Zimmermann (Germany). The ProMath Group was founded in 1998 on the suggestion of Erkki Pehkonen (Finland) as a Finnish-German Group.

The ProMath Group wanted to promote the research about problem orientation in mathematics teaching and its practice in school. In July 1999 this Group met in Jena (Germany) on a conference about creativity. In September 2000 they organized a little meeting at the University of Bielefeld (Germany). In May 2001 they met in Turku (Finland), where a Greek Colleague joined them, so the ProMath Group became an international group. In September 2002 a meeting took place in Bielefeld. In 2003 a ProMath Conference was held in Jena with participants from Denmark, Finland, Germany Greece and Hungary. In June/July 2004 a bigger conference was arranged in Lahti (Finland). This conference was before the ICME 10 (Copenhagen), so a lot of people could visit it. In September 2005 the sixth ProMath Conference took place at the University of Debrecen (Hungary). The seventh ProMath Meeting is planned to take place in September 2006 in Komárom (Slovakia).

Proceedings were published for the following meetings: Turku (2001), Jena (2003), Helsinki (2004) and Debrecen (2005).

Keywords and phrases: Problem solving, new researches on teaching of problem solving, problem solving processes, mathematically talented pupils.

ZDM Subject Classification: C 30, D 50.

Debrecen, May 2006

Tünde Kántor

Introduction to the sixth ProMath-Conference

Dear colleagues and friends,

working with problems in mathematics teaching is very important for reaching several general aims, for the students view of mathematics and so on. I will not go in details here; you can find a lot of literature about it. Though this is well-known in didactical theory the practice at school often is missing such phases of problem orientation.

To promote the research about problem orientation in mathematics teaching and its practice in school on the suggestion of Erkki Pehkonen in 1998 we did found a Finnish-German-Group which later on was called "Promath-Group".

In July 1999 this group met in Jena together with mathematicians and psychologists on a conference about creativity[1]. After that in September 2000 the ProMath-Group arranged a little meeting at the University of Bielefeld. In May 2001 then we met in Turku where a Greek colleague joined us[2]. From that time the ProMath-Group slowly change from a binational to an international group. In September 2002 another little meeting took place in Bielefeld. One year later in September 2003 a ProMath-conference with participants from Denmark, Finland, Germany, Greece and Hungary was held in Jena[3]. In June/July 2004 the ProMath-meeting was arranged as bigger conference in Lahti just before the ICME in Kopenhagen. So a lot of people out of the whole

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[1] For the proceedings of the conference see: Zimmermann, B. et al. (eds.) 1999. Kreatives Denken und Innovationen in mathematischen Wissenschaften, Jenaer Schriften zur Mathematik und Informatik Math/Inf/99/29.

[2] See: Veistinen, A.-L. (ed.) 2002. Proceedings of The Pro Math Workshop in Turku, University of Turku, Department of Teacher Education, Pre-Print nr. 1, 2002.

[3] See: Rehlich, H. & Zimmermann, B. (eds.), 2004. ProMath Jena 2003 – Problem Solving in Mathematics Education, Franzbecker: Hildesheim.

world which are interested in problem solving in mathematics teaching could visit this conference[4].

And now I am glad that we can hold our sixth ProMath-conference in Hungary where problem orientation in relation to mathematics has as long tradition. But I am also glad to be here in Hungary because I can see again several Hungarian friends.

I would like to send big thanks to all local organizers, especially our colleague András Kovács, for helping us to prepare and arrange this conference. I am sure we all together will have a nice and successful meeting.

Günter Graumann

[4] See: Pehkonen, E. (ed.) 2005. Problem Solving in Mathematics Education – Proceedings of the ProMath meeting June 30 – July 2, 2004 in Lahti, Department of Applied Sciences of Education, University of Helsinki, Research Report 261, Helsinki.

Space perception abilities (and space geometrical problem solving) in a group of 3rd year university teacher students

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Abstract: *In my PhD research I focus on teaching space geometry and space geometry problem solving combining different tools (computer animation, traditional models, worksheets etc.). I work with third year university teacher students. In this article I write about a pilot study pretest, especially about its space perception part. With help of this test I try to analyze students' space perception abilities and space geometry problem solving skills, which are relevant to my work. With help of this test I try to identify their level in these skills.*

Introduction

In school geometry carefully designed demonstration is very important, because through concrete experience students learn to collect and acquire attributes and to develop and formulate a mathematical concept step by step. During this process students have to separate (differentiate) the individual and general, static and dynamic, concrete and abstract character of the used models. Within this process are necessary consciously designed, demanding and various models.

At Eötvös Loránd University I run a course with third year university students who want to become teachers of mathematics whose title is "demonstration and experimentation on math lesson". In these lessons we look for typical school teaching situations. The central aim of these lessons is to give personal experiences in addition to the theoretical discussions.

Usually, we choose a problem for university students; we prepare a set of computational and paper and pencil tools, which can help them to find the solution. During the problem solving process they can choose the way of visualization (enactive, iconic and symbolic in sense of Bruner (1970)), the method of solution and the social form. I try to show a possible design of a demonstration relevant to the problem. The students have to transfer this method to the secondary school level. To transfer this method means to formulate a relevant problem, considering the proper knowledge, the way of thinking and the ways of learning of the pupils. By the preparation of the problem solving process they have to collect different tools for designing a demonstration. These exercises give each student some didactical experience depending on their problem solving skills (see more in Berta (2005).

My main research focus is on the question how to combine traditional teaching and computer based tools (three basic tools: worksheets, manual model and computer animation) in mathematics education, mainly in teaching of space geometry. I made a pilot study to prepare a main case study with my students in a „demonstration and experimentation on math lesson" course.

My main research questions in this case study were:

What kind of changes happen in teacher student's (pedagogical) knowledge, when they have worked through the course "Experimentation and modeling on mathematic lesson":

1. *in the use of space perception skills*
2. *in the understanding space perception skills*
3. *in the space geometrical problem solving skills?*

Pretest on space perception

Visualization and primary experiences should play a more important role in geometry, in order to highlight deeper mathematical relations and to make the process of understanding easier. Without a complex use of space perception the teaching and understanding of higher-level space geometry becomes useless as skills for generalization, problem-understanding and problem-solving cannot be developed properly.

As a part of this pilot study I gave a pretest to my student at the beginning of the course. This test was specially designed to find answers to the following questions:

1. What are the students' own attitudes towards computers before beginning the course?
2. What is the students' opinion about computer aided mathematics teaching?
3. What is the students' opinion about model aided mathematics teaching?
4. What is the students' opinion about the importance of geometry teaching?
5. Which intuitive space geometry skills and space perception ability do students have?
6. Which skills of space geometry problem-solving do the students have, what strategies do they use?

In addition to the outcomes of the test I tried to find out explanations, which might help me to elaborate the problem-solving strategies used by the students and their way of thinking.

The test had three parts:

- A questionnaire part – including 16 questions, by which I tried to find possible answers for questions 1-4.
- A space perception part – including 7 tasks – in details in the next part of the article (cf. pp. 6.).
- A space geometry problem solving part – two space geometrical secondary school tasks, including the possibility for using resp. requiring different problem solving strategies and space perception abilities (cf. pp. 5-6.).

By this test I wanted to figure out what kind of ability my students have and how they can solve space geometrical problems. By the analysis of their works I can properly prepare developing lessons of the course. In the following part I will write about the space perception part of the test and its evaluation.

Theoretical background

We can find several relevant theoretical works about visualization. These include description of some keywords as visualization, mental image, spatial ability, visual imagery, etc. which are important to my work as well.

Space perception and imagination and mental representation of geometry results are complex (to see, to recognize, to manipulate, to describe, to re-structure, etc.) (brain) activities with help of mathematical knowledge's, skills, abilities.

Dreyfus (1991) said that in mathematics education a spatial visualization is a set of elements related to the generation and use of mental representations (mental images) of mathematical information. This set of elements integrating spatial visualization can be divided into three main parts: mental images, visualization and visualization abilities.

I consider ***"visualization" in mathematics*** as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties. Visualization is integrated by four main elements: *Mental images, external representations, process of visualization* and *abilities of visualization.* (Vásárhelyi 2001, 2002; Dreyfus 1991; Presmeg 1968; Yakimaskaya 1991 and Gutierrez 1994)

Mental images are any kind of cognitive representation of mathematical concept or property by means of visual or spatial elements. They are the basic objects for spatial visualization and imagination. Some types of mental images are described by Presmeg (1986): concrete pictorial images, pattern imagery, memory images of formulae, kinaesthetic imagery and dynamic imagery.

External representation is any kind of verbal or graphical representation of concepts or properties including pictures, drawings, diagrams, etc. that helps to create or transform mental images to the visual reasoning.

Process of visualization is a mental or physical action where mental images are involved.

Bishop (1989) has found out the necessity of separating knowledge of the problem's content and representation of it from the abilities required to perform successfully in that context. He described two kinds of ability:

- The ability for *visual processing (VP),* as a process which involves the ideas of visualization, the translation of abstract relationships and non-figural data into visual terms, the manipulation and extrapolation of visual imagery, and transformation of one visual imagery into another.
- The ability for *interpreting figural information (IFI),* as a process of reading, analyzing and understanding spatial representations in order to obtain some data from them.

The fourth component of visualization is constituted by the ***abilities*** which help to carry out the previous processes. The learning and improvement of these abilities is the key in the whole process of spatial visualization (Gutierrez, 1996b, 1996c). Depending on the mathematical problems to be solved and the images created, students should be able to choose among several visual abilities. These abilities may have quite different foundations. Del Grande (1990) compiled these abilities. Some of them are relevant to my research:

A1. *Figure – ground perception* - the ability to identify a specific figure by isolating it out of a complex background.

A2. *Perceptual constancy* - the ability to recognize that some properties of an object are independent from size; color the ability to texture or position and to remain unconfused when an object or picture is perceived in different orientation.

A3. *Mental rotation* - the ability to produce dynamic mental images and to visualize a configuration in movement.

A4. *Perception of spatial positions* - the ability to relate an object, picture, or mental images to oneself.

A5. *Perception of spatial relationships* - the ability to relate several objects, pictures, and/or mental images to each other, or simultaneously to oneself.

A6. *Visual discrimination* - the ability to compare several objects, pictures and/or mental images to identify similarities and differences among them.

Spatial processing ability is the ability needed to fulfill the combined mental operations required to solve a spatial task. It includes not only the ability to imagine spatial objects, relationships and transformations and to develop them visually, but also the ability to solve the tasks using the ability to encode them into verbal or mixed terms (Gorgorió, 1996, 1998).

Spatial processing ability includes:

- At least as many different abilities (see before)
- **The ability to interpret spatial information**
- **The ability to communicate spatial information**

Description of researched group

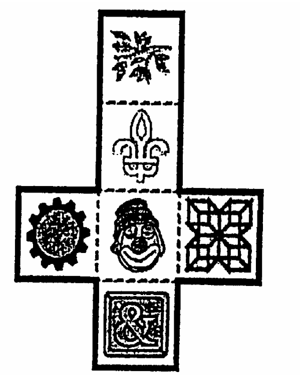
In the researched group there were 15 third year university math student teachers who enrolled the course “demonstration and experimentation on math lesson”. For them this subject was the first meeting with didactics of mathematics during their studies at the university. They were rarely at the same level in geometry – having secondary school level and one semester of university geometry finished.

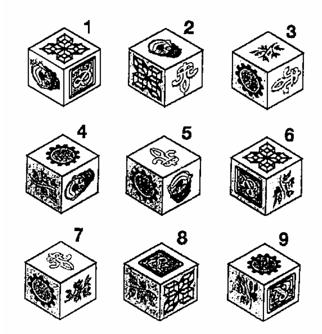
Description of the “Space perception part” of the pretest

In the test I prepared 7 different tasks which should measure different abilities. (After the course we wanted to have a similar posttest again to get information about the improvement of these abilities). In each task students not only had to solve them, but they had also to write down the way how they had solved it. With the help of their descriptions it was easier to analyze their solutions, and it was very important to see how they could interpret and communicate spatial information. As prospective teachers it is very important for their professional life to acquire these abilities, because they should not only be able to solve tasks like these, but they have also to teach it to their students. With consciously planed teaching we could improve space perception abilities and interpretation and communication of it. The difference in people’ skills and abilities – space perception too – depends not only on their natural born attributes, characters, but on the whole developing process as well.

When analysing the students´ solutions, I looked not only at different abilities, but also if the solution method they used was *visual* or *non visual* and if they looked at the problem *globally* or *partially*. I interpreted the solution to be *visual* if the student had used visual images as an essential part of the method of solution and *non-visual* if the student had used an argument without relying on visual images while solving the task. The strategy was *global*, if the students´ cognitive strategy had focused on the object considered as whole; and *partial* when student had focused on some parts of the object.

Task1: *Which cube can be fold from the left cube-net? Sign on the net where you want to make strips for gluing, and which edges belong to each strip.*



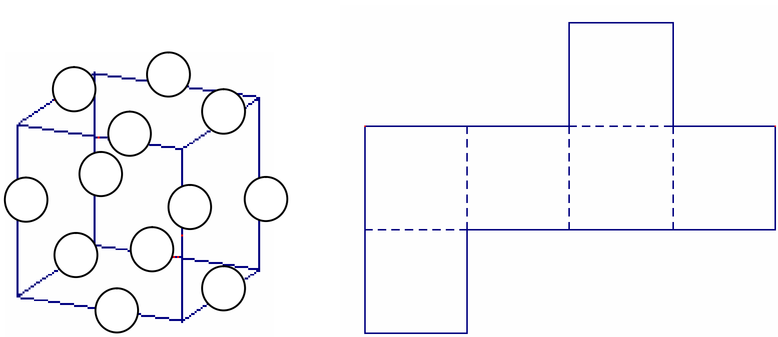


1 ...., because...
2. ...., because...
3 ...., because...
4. ...., because...
5 ...., because...
6.........., because...
7 ...., because...
8. ...., because...
9 ...., because...

Which edges belong to each strip?

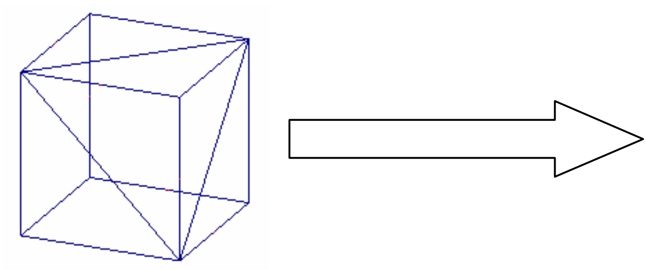
Task1 focuses on abilities A2, A4, A5. Student had to make the cube mentally from the net of the cube (from plane to space) and find out the different rotation of figures on that side. It was a task where most of the students (10 of the 15) found out right answers. There were problems with describing of that solution. Seven students could write an answer for every cube. When solving this task all of the students had chosen a visual strategy, they imagined how to fold this cube and after it they checked the figures on the presented paper and compared with drawn ones. They looked on it just from sides like they saw at the pictures. Nobody cared about the sides which we could not see on pictures of the cube. Nobody thought about the figures of those sides.

Task2: *Write numbers at the corresponding edges left and right which make clear, how to cut the cube if we want to get after cutting the net on the right side!*



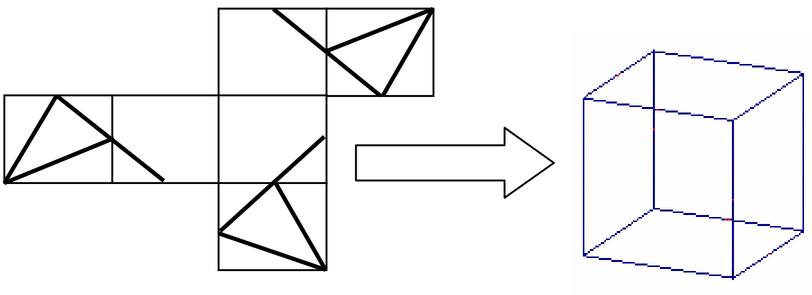
To some extent task2. is an opposite task to task1. This task 2 (?) focuses on abilities A2, A4. We have a cube and out of this the students had to make a given net. As a help there were given reasons how to start - with numbering of edges. 14 of 15 students used this numbering. One of the students solved this task by denoting the vertexes. It was interesting how they described their solutions. Everybody made it correctly, but not everybody could write down the way of solution. 8 students wrote nothing. Three students out of 7 who described their solution started to write numbers on edges of the cube-net first and after this they put numbers at the edges of the corresponding cube. When solving this task they applied the same thinking style and transfer from plane to space as in task1. In their description of solutions they used non-visual methods (just numbering and use of letters of the alphabet (nobody wrote that he or she imagined the process of cutting in their mind) and worked with the cube as a whole – globally. The solution of this task showed that there had been a big problem with interpretation.

Task3: *We truncate the cube in a way you see in the axonometric presentation of the cube. Make the net of the truncated cube. Explain your solution!*



Task3 focuses on abilities A1, A2, A4, A5. Students had to imagine the truncated cube mentally, had to find out these two polyhedrons which we get after cutting. It was a hard task for them. Just 3 students tried to write down the way of solution. From their drawings we could see that most of them used non-visual methods to solve it. This could also be derived from their written description of their solution. First they drew a net of the cube, later on they identified the vertexes of the cube with vertexes on a net and in this was they could draw the net of truncated cube, too. Now I saw that some students had problems with the meaning of nets, but 10 drawing out of 15 were correct.

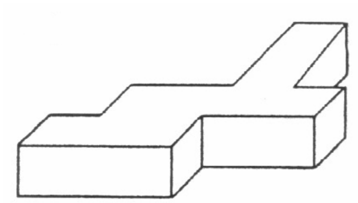
Task4: *Draw into the axonometric figure of the cube the section which is represented on the net. The endpoints of the cutting sections are midpoints of the sides or vertexes. Explain your solution!*



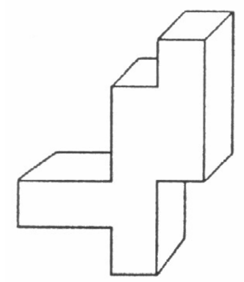
Task4. is an opposite task to task3 (from plane to space). It focuses on abilities A1, A2, A4, A5. There was nobody who would make it visually, all students who solved this task gave numbers to vertexes on a net first and after it identified this numbers with the vertices of a cube and with help of them they draw cutting sections into the picture of the cube. There were just 3 students who could write down their solving process. 3 out of 15 students did not try to solve this task. From 12 solvers there were 4 who made mistakes in identifying vertices and therefore they did not find the right solution.

Task5: *Which of these four pictures represent the same solid? Prove your answer!* (by Gorgorió)

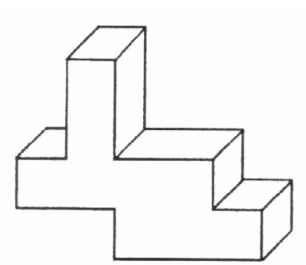
A)



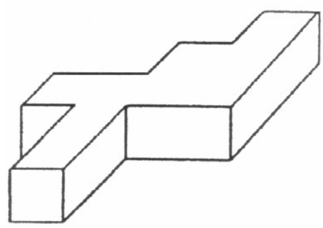
B)



C)

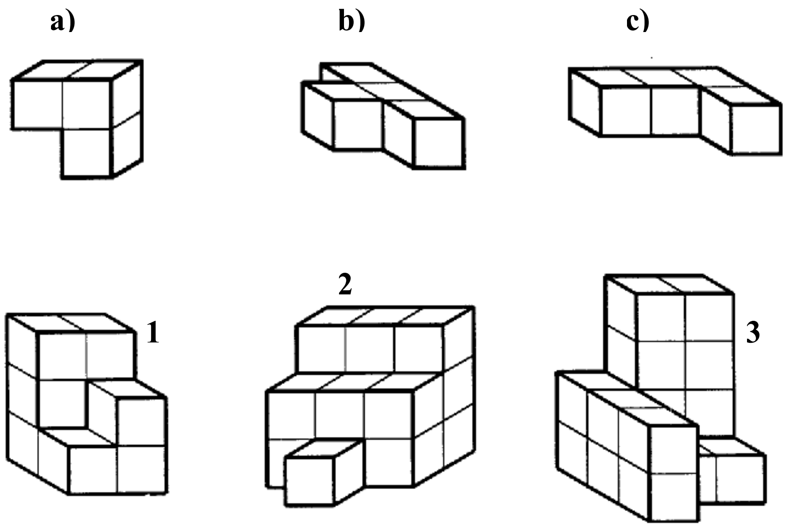


D)



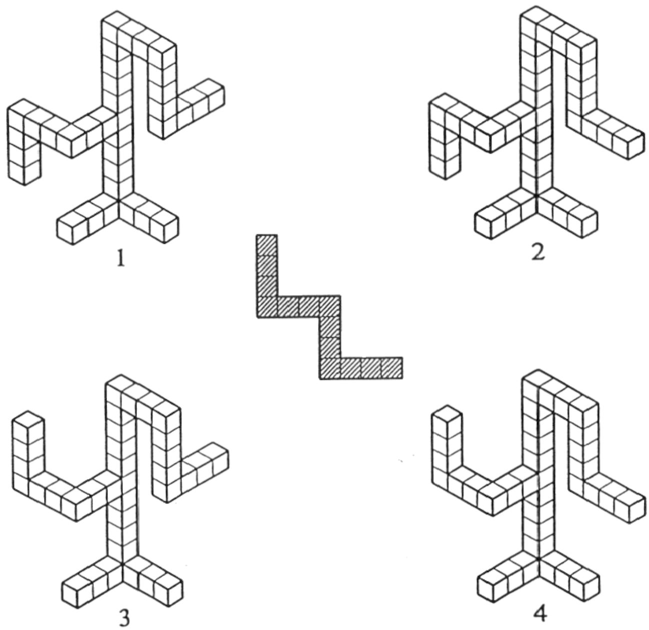
Task5 focuses on abilities A2, A3, A4, A5 and A6. Students had to use mentally rotation. Most of them found it and tried to do it. Two of them drew into pictures Cartesian coordinate system for better seeing the way of rotation. Everybody tried to solve it, they found the good answers but they were not able to write down there way of solution. Not everybody used rotation in that task. Some of them divided the solid into unit cube peaces and they counted how much peaces are in each polyhedron. There were also students who made mentally rotations and then - for being sure in a right solution - they also divided these polyhedrons into unit cube peaces. Everybody solved this task correctly.

Task6: *Which of the solids 1, 2, 3 can be constructed with help of a, b, c? Explain it!*



Task6 focuses on abilities A2, A3, A4, and A6. My main interest in task was determined by the question whether the students could imagine also another view on solids 1, 2 and 3 during their solution process. Nobody described his solution and only two out of 15 students thought also about a view from the back side. There were 3 students who could not find the solution for any of the 3 solids.

Task7: *Which of the four solids – presented by figures in the four corners of the whole drawing - can be moved in such a way that its projection into the plan corresponds to the picture in the center? Explain your answer!*



Task7 focuses on abilities A2, A3, A4, A5 and A6. It was also about mental rotations of solids and about different views on it. For the solution it was important to get an overview. Not everybody could make it and imagine the rotation. Some of them tried to solve it by counting the unit cubes and comparing the directions of that. Nobody could write down the solution clearly, just 6 of them tried to write it down. Everybody found the right solution.

Conclusion

We can see from the solution of the students that their ability to interpret and communicate spatial information is on a very low-level. It is important to improve more these two abilities by lectures than the other abilities. It is very important for our student teachers to try to verbally describe their process of thinking and solving of tasks and problems. It is also very important to work more with polyhedrons and with different representations of it (concrete models, pictorial, computer models etc.) to get a more comprehensive experience in working with them and to get better and correct mental images. As a second part of my research I prepare with help of this pretest and pilot study corresponding lectures to help my students to improve these abilities. In a last phase of this research I want to administer a posttest to them to see the changes in their abilities. I would like to compare each person's pretest and posttest and analyze the change. I experienced that the short and not disappointing lectures had motivated the students. The students preferred that subject that helped them to teach. In my pilot study I could observe this motivation and interest.

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Three Main Ways to Improve the Instruction in Mathematics

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Abstract: *The demand for more elements of problem solving and higher-order thinking in mathematics instruction is widely discussed nowadays. This article suggests three main ways to improve the instruction in mathematics in secondary schools in order to meet these demands. The three ways are using questions and inductive reasoning, using problem tasks and using investigations and modelling projects. Some examples from the author's collection of research data in a Finnish secondary school are used to elaborate the issue.*

Demands of improvement for mathematics instruction

Since Polya (Polya, 1945), a strong demand for more elements of problem solving in mathematics instruction has been growing. It has taken the form of both large movements and different kinds of statements from individual researchers. The huge process in the USA leading to Principles and Standards (NCTM, 2000) is an example of the former kind and Lenni Haapasalos (Haapasalo, 1995) course book about problem-solving processes is an example of the latter kind.

Moreover, the development of special conferences in problem solving in mathematics, ProMath conferences, can be seen as a result of the same concern among researchers in mathematics didactics.

Demands for real problem solving and higher-order thinking are widely discussed and for instance in Proceedings from ProMath in Jena 2003, Erkki Pehkonen (Pehkonen, 2004) states that problem solving commonly is accepted as a tool when the aim is to develop thinking skills, Henry Leppäaho mentions problem-solving strategies (Leppäaho, 2004) and Pál Maus (Maus, 2004) asks for more emphasis on the type of thinking needed in problem solving in mathematics instruction.

In the ProMath conference in Lahti 2004, Kaye Stacey (Stacey, 2005) demonstrates that "problem solving has become difficult to pinpoint in the official descriptions of mathematical curricula in some countries because it has become a part of a pervasive socio-cultural approach to learning mathematics". With examples from the current mainstream approach in English-speaking countries like Australia, the UK and the USA, she finds a mix of goals for process aspects. Such goals could be entitled "Using and applying mathematics" and include problem solving, communication and reasoning (to choose the UK variant of saying it). Especially, she mentions open problem solving as a process through which students could learn.

Also in Lahti Günther Graumann (Graumann, 2005) claims that the aim of the conference is to "strengthen problem orientation in mathematics education" and suggests that students more often should work with problem fields of everyday life. He even argues that if roughly one third of the mathematics education should be spent on mediation and information and another third should be spent on deepening knowledge in exercises and applications, then the last third should be reserved for investigations and working on problems. According to him, working on problems could mean working as individuals, in pairs or in groups, but sometimes also the whole class can work together on a problem.

Ways to improve the instruction in mathematics

What should a common teacher in an ordinary class do to meet these demands for more elements of problem solving? Moreover, what can be done when the teacher constantly feels that there is a severe lack of time? As follows, using some examples from my collection of research data, I am going to suggest that the answer consists of mainly three kinds of efforts: the use of questions and inductive reasoning for the teacher and the use of problem tasks as well as investigations or modelling projects for the students to work with.

Questions and inductive reasoning

When preparing the lessons and later when acting in front of the class, the teacher needs to be aware of the importance of activities like investigation, reasoning, making conjectures and testing of hypotheses. No matter what the topic is, it is probably possible to use one or several of such elements of problem solving. The teacher can pose challenging questions and use an inductive kind of reasoning in order to make the students themselves find out how to proceed in different situations when the teacher is presenting unknown methods or when the teacher and the students are working together with a task.

The task in the following example has been used in ordinary secondary-school classes with 15 year old pupils. They have worked with the task individually without discussion in their classes. Of course, there is also the possibility of inviting the pupils to work with similar tasks in small groups.

First example

Some variable terms are going through “a procedure” which gives new variable terms as the result. The results of the procedure in four cases are the following:

x^6	gives	$6x^5$
x^{12}	gives	$12x^{11}$
$3x^7$	gives	$21x^6$
$5x^{20}$	gives	$100x^{19}$

Decide what happens when the same procedure is applied to the variable terms x^4, $7x^5$, kx^n and the somewhat special variable term $25x$.

It is obvious that the pupils learned how to find the derivative but this was not the point, although a positive interest in coming mathematics arose in the class. Their challenge was to find the pattern that described a procedure of a kind they had never heard of or seen in a textbook. In my first test class, all the pupils managed to get the correct answer in the case kx^n, which was a very good result.

In the actual case, the students only got the message that some of them most probably will meet the same procedure later on and then make very good use of it. The same kind of task has also been used when introducing the concept of derivative in a short course of mathematics in upper-secondary school and that was done with a considerable success. It then became a source of motivation for the students and as a result they could perhaps understand more of the theoretical background and the use of the procedure.

Problem tasks

Problem solving is often practised with tasks, in which a certain problem-solving strategy is intended to be used and sometimes the tasks may have a context from the real world outside school. I have also used problem tasks which invite the students to use heuristic methods and different problem-solving strategies. If it is possible I use tasks with a real-world context because then the task is even more valuable and can encourage the students to make connections to the real world outside school and help them to apply their knowledge in mathematics to different kinds of real situations in the future.

The following example was also used in a class with 15 year old pupils, but this time, the pupils were supposed to work in groups of about four pupils. The problem was well connected to the actual topic in mathematics (applications with quadratic equations where square roots are needed) and a period of cooperative learning, although it was not a task that could be classified as a good simulation of a real-world task. More about classification of problem tasks can be found in Palm & Burman (2003).

Second example

A group of maths students was going for a (mathematical) walk in Lapland. They decided to walk some whole number of kilometres west, then some whole number of kilometres south and finally, the shortest way to the starting point. The task is to find the two whole numbers (it may also be the same number) in order to get a total of 20 km or as near that distance as possible.

When this task was carried out, it caused nearly half an hour of intensive work. The groups worked hard to find the best answer and it was also said that the fastest group to find the best answer would get at least some kind of honour award. The best answer was also found, but it was possible for the group to know that it was the best answer only after having tried all the combinations of two possible numbers. Thus, there were different possibilities even if the group had the right answer:

- a group could have the right answer without being sure of having it
- a group could have the right answer and also know it

- a group could have the right answer and also know it but the group did not need all the time available because of organizing the work in the group very well

Of course there is also the possibility that a group fails completely, perhaps as a result of bad organization or as a result of not having the capability to solve the problem at all. This raises the question of how to form the groups, but that problem will be left aside in this article.

The next example has been used in several classes with about 16 year old upper-secondary school students and they were supposed to work individually.

Third example

The average of seven different positive whole numbers is 23 and the median is 20. Find the greatest possible number in the set.

This task seems to be very simple but nevertheless, it requires knowledge of two concepts and a correct conclusion about every single number involved.

Investigations and modelling projects

The example above can be very good training in problem solving but it is certainly not the kind of problem that we frequently meet in the real world. There is a need for practise in working with real-world situations and dealing with problems related to the real world. Questions and problems from real-world situations can seldom be solved in one lesson or two. Therefore, we need another kind of task: for pupils in lower-secondary school they could be called investigations and for students in upper-secondary school modelling projects.

However important real-world tasks are, an investigation can also be an investigation in mathematics without a real-world context.

The next example is an investigation without a real-world context, but a very basic and useful investigation into mathematics. It has been used as a model for investigations, at the beginning of the seventh grade with 12-13 year-old pupils.

Fourth example

Write down all the numbers from 2 up to 100 beneath each other and then fill in the prime factors of all these numbers like for instance $6 = 2 \cdot 3$, $7 = 7$ and $8 = 2 \cdot 2 \cdot 2$. Try to investigate some strategies that can help you to find the factors when you are proceeding up to 100.

As this one might be the first investigation made by the pupils, the teacher and the pupils have often worked together up to 20. Then the pupils can go on in small groups up to 50 or 80, depending on the time available and the pupils' capability. The last part can be done as an individual work or as a work in groups and then it is important to find strategies for factorizing new numbers.

One of these rules might be that every even number has the factor 2 and also all the factors of the number we get when the even number is divided by 2. Another rule might be that every seventh number has the factor 7, every eleventh number the factor 11 and so on. In one class some pupils even came up with the question how to know when it is no longer needed to check if a certain number is a factor in the actual number. As an answer they found what could be considered a description of a square root, although this concept was not yet known.

The final example is taken from upper-secondary school and the actual group of 18 year-old students had chosen a long course in mathematics, which means ten courses of about thirty lessons of 45 minutes each in Mathematics. The example is chosen from the course called Probability and statistics and in this case the students were supposed to work in groups of about four students. They could choose a problem area and because of the limited time, the teacher helped the groups to formulate problems which had a suitable length and a reasonable connection to the course. The groups should also have different problems.

Fifth example

In what month are the students in grade 11 at our school born?

This question was raised by some students in the class and accepted by the teacher to be an interesting and quite suitable question to work with. The students also immediately had the needed hypothesis ready: throughout the whole school system those born at the beginning of the year have a kind of advantage which might give the outcome that the majority of the students in the upper-secondary school could be born in one of the first six month of the year.

With some help of the teacher, the actual group of students was able to collect data and compare it to a uniform distribution and then they drew conclusions from that. The two final steps were to explain the results and compare them to the hypothesis and thus, to evaluate the model and make a suggestion how to improve the model. After each step in a modelling project like this the groups gave the teacher their reports and got feedback. If there is time, it is of course recommendable for the groups to present their results to the whole class.

Discussion

We have seen examples of the use of questions/inductive reasoning, problem tasks and investigations/modelling projects in mathematics instruction. The setting is a Finnish secondary school and moreover, the author has tested all the tasks in classes where Swedish-speaking students receive instruction in Swedish.

The research methods have been action research (first part, lower classes) and didactical engineering (second part, upper classes).

In the Finnish school system over all and especially in upper-secondary school, there is a strong feeling that the time given to mathematics is very restricted (and much more time is given to languages). Accordingly, the chosen tasks may reflect that the teacher cannot use much time for them, when there in every course is a time-consuming content, which after three years in upper-secondary school is tested in a matriculation examination. Nevertheless, the examples show a desire to take steps in the supposed right direction, although the steps are small steps. Above all, the article gives hints of what a teacher can do in order to improve the instruction in mathematics even in a very time-restricted system.

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Mathematics and Music

A Topic for Interdisciplinary Problem Fields

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Summary

Music offers a lot of opportunities for connections of mathematics and arts on secondary level. First of all rhythm and notations of classical music provide exercises for fractions (in grade six or seven) at which we also can deepen knowledge about music. The analysis of different chords and the sounds of instruments combined with creating superpositions of sinus-functions then build a problem field for grade ten or eleven. Also the analysis of number-symbols and geometrical forms within the notation as well as symmetric patterns in given compositions or the discussion of structures of modern composition techniques are problem fields for connecting mathematics and music on higher secondary school level up to university level. The development of pitches and scales respectively tunes is another problem field in which the development musical theory from Pythagoras up to the twelve-tone-technique can be opened up by mathematics.

In my presentation I will schedule such problem fields concerning mathematics and music and go more into details with the problem field of the development of our heptatonic scale of tones in Pythagorean, diatonic and well-tempered tune.

Introduction

On earlier meetings of the ProMath-Group and also on other conference in the last five years my focus for problem orientation in respect to mathematical content was lying on geometry. Last year in Lahti then I concentrated on another aspect - namely mathematics for everyday life. The aspect of practice orientated mathematics education as well as application orientating and modelling in mathematics lessons is in my research interest – so as geometry and didactics of geometry – already since three decades.

This aspect of using mathematics for outer-mathematical problems I will widen here by looking on a combination of mathematics with another discipline respectively different subject at school whereat today I will focus on mathematics and music.

I did choose this combination because on one hand mathematics and musical theory have a common basis with Pythagoras as well as influenced each other. On the other hand music belongs to my interests and both mathematics and music do fit very well with Hungary as far as I know.

Mathematics and Music – a survey of different problem fields

First of all within the combination of mathematics and music there is the theme “rhythm”. In music of course rhythm is very fundamental. For example elementary music education basing on Kodaly (which is practiced not only in Hungary) starts with different exercises in rhythm. Moreover rhythm is structuring popular music as well as classical music. But rhythm also is a medium for structuring in mathematics. From J. Kühnel for example we know that rhythmic counting is a good way to structure the field of natural numbers. A problem field combining mathematics and rhythm in music concerns the notation of classical music and its different measures. Up from grade six we can analyse by counting and adding fractions given notations of music or different distributions of notes within a given measure. As an exercise for example we can let the students make time lines in notations of a melody without time lines or complete notes or break sings in incomplete notations. Also we can find out that in music a whole time-unit not always is equal to a whole number.

A second problem field concerning mathematics and music has to do with the tuning respectively the definition of intervals of tones. For occidental music the basis for this is - as already mentioned - with Pythagoras where mathematics, musical theory and religion have been a whole. I will focus on this theme a little bit more at the end of my presentation.

Very interesting for students of higher secondary school is the analysis of sounds as well as of special instruments. With this we can combine working with trigonometric functions with music as well as physics (especially acoustics). At first we find out that a periodical oscillation can be modelled with a regular rotation on a circle where the graph leads us to the sinus function. After that by analysing overtones of the sound of an instrument we are lead to different super-positions of sinus-functions. Also we look at the graphs of super-positions for harmonic or not so harmonics sounds as well as floatings with disharmonic sounds. If we can get a synthesizer from the music teacher we also can create new sounds respectively new instruments. Finally we can make a mathematical analysis of the family of all possible super-positions of sinus functions respectively cosines functions.

The other way round we can transform different waves known from physics (for example gravitation waves or electromagnetic waves) into acoustic waves and represent as sounds. Also the so-called music of planets from J. Kepler in his “harmony of world” where the ratios of distances respectively time of rotations are transformed into tones can be seen as musical representation of a physical or mathematical phenomenon.

Another field concerning mathematics, music and acoustics has to do with reflections of different tones in special designed rooms. We for example know about the good acoustic in ancient arenas. With students of upper secondary schools we can find out that in an arena with a shape of an ellipse a whistling in one focus can be heard in the other focus. But also the time difference of a sound and its resound in different halls with special geometrical shapes we can compute with students.

Furthermore we find problem fields concerning mathematics and music by analysing classical compositions with help of numbers, symmetry or geometry. Numeral symbolic we can find in several compositions of J. S. Bach. A generalized symmetry we often find in symphonies and very well known are inverted forms of a melody (changing up-going intervals into down-going intervals of same size and the other way round, let run the melody reverse as well as combine both) or the translation of a melody within a canon or fugue. Moreover we can find special patterns (e. g. a melody going up combined with a melody going down as symbol for the cross of Jesus Christ). With the composition-program PRESTO you can analyse and transform compositions with methods which are similar to those of dynamic geometry software[1]. PRESTO originally was made for Atari-computers but with an emulator it runs under Windows. It is based on the finite Galois-field of 71 numbers. Within the 71×71 pixel-field you can change between demonstrating

[1] See e.g.: Christmann, N. (2005). Dynamische Geometrie und Musik. In: Beiträge zum Mathematikunterricht 2005 (Proceedings of the 39th Conference of Didactics of Mathematics 2005 in Bielefeld), Hildesheim, p. 145-149.
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Compare also: Leopold,, C. (ed.) 2003, Klangansichten – Musik sehen – Geometrie hören, Technische Universität Kaiserslautern (ISBN 3-936890-19-6)

the graph of amplitude and time or duration and time or loudness and time of a melody given in.

Similar to the field just named is the investigation of compositions of twelve-tone-technique. For example the analysis of different trails within the twelve-tone-circle can lead to finite cyclic groups and its subgroups. But still the combinatorical question about the number of all possible twelve-tone-units can be interesting.

More difficult then is the analysis of other modern music which for example uses clusters or special wavelets. Also the analysis of computer generated modern music might be interesting for students of higher secondary school.

There are a lot of other problem fields combining mathematics and music but because time goes on I now will go into more details about the history of tuning of our classical music.

Tunings of tone-intervals and scales in occidental music

The oldest well-known interval is the Octave. Presumably it was used already very early in the history of mankind because men and women normally sing parallel in Octaves if they sing the same song.

A subdivision of this interval in five steps (called Pentatonic) was used in many melodies of ancient music. Between 1000 and 500 BC in Egypt a system with ***seven steps within an Octave*** (called Heptatonic) was developed. We have to presume that Pythagoras became acquainted with this system during his travels to Egypt. In any case we know that ***Pythagoras*** used a subdivision of the octave with seven steps. This turned into the basis for the whole occidental music.

Pythagoras demonstrated the intervals on the so-called monochord (a wooden resonance box with one string which can swing in different length). The

ratios of length he used as ***description of the intervals***. So e.g. the octave is defined by the ratio 2 **:** 1 because the length of the string for the Octave is half as long as the length of the string of the basic tone (the Prime).

If you are looking for piling up Octaves you easily can find out (and hear at the monochord) that the Octave of an Octave (Double-Octave) is characterised by the ratio 4 **:** 1 while the ratio 8 **:** 1 is belonging to the Triple-Octave, the ratio 16 **:** 1 to the Quadruple-Octave and so on. With this we find out that the ***"addition" of intervals***[2] corresponds to the ***multiplication of the ratios*** and a "multiplication" of an interval corresponds to a power of the ratio.

As basis of this **Pythagorean scale** it was used the third overtone with ratio 3 **:** 1 respectively its Octave-transposition with ratio ***3 : 2*** called ***Fifth*** or ***Quint***. By piling up and down this interval (and using Octave-transpositions if necessary) Pythagoras defined his seven-step-scale of an octave with following ratios:

Prime (basis) → 1 **:** 1

Second (Tone) → $(3 : 2) \cdot (3 : 2) : (2 : 1)$ = ***9 : 8***

Pyth. Third → $(3 : 2)^4 : (4 : 1)$ = ***81 : 64***

Quart → $(2 : 1) : (3 : 2)$ = ***4 : 3***

Quint → 3 **:** 2

Pyth. Sixth → $(3 : 2)^3 : (2 : 1)$ = ***27 : 16***

Pyth. Seventh → $(3 : 2)^5 : (4 : 1)$ = ***243 : 128***

Octave → 2 **:** 1.

[2] We mostly see musical intervals as straight lines which can be added. This e.g. is obvious with the keyboard of a piano or an organ.

As difference between one interval and the next one we easily can find by computing division of fractions the whole Tone (Second) [3] with ratio 9:8 and the so-called ***Half-Tone*** [4] with ratio ***256 : 243***.

Because the ratios of the Pythagorean Third, Sixth and Seventh as well as of the Half-Tone are not so simple as the others already in the first century BC there was introduced by Didymos a new tuning with only little changes which later on was called "**diatonic or harmonic scale**". Sometimes it is also used the name "pure tuning". Didymos used for the ***Third*** instead of 81:64 the ratio 80:64 which equals ***5 : 4***. As consequence we then get a bigger and a smaller Second as well as changes for the Sixth, Seventh and the Half-Tone[5].

With Pythagoras the Third could be seen as addition of two Seconds (Tones)[6]. But because the diatonic Third is a little bit smaller as the Pythagorean Third the difference between the Third and the Second is also a little bit smaller than the normal Second (Tone). This so-called ***Reduced Second*** has also a "good"[7] ratio, namely ***10 : 9*** (because of 5/4 **:** 9/8 = 10/9). The diatonic Sixth with ratio 5 : 3 then can be seen as addition of a Quint and a Reduced Tone (3/2 • 10/9) as well as addition of a Quart and a Third (4/3 • 5/4) and the diatonic Seventh with ratio 15 **:** 8 can be seen as addition of Quint and Third (3/2 • 5/4) as well addition of a Sixth and a Second (5/3 • 9/8). The **scale of the diatonic tuning** therefore is the following:

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[3] E.g. as difference between Quint and Quart corresponding to (3:2):(4:3) = 9:8.
[4] E.g. as difference between Quart and Third corresponding to (4:3):(81:64) = 256:243.
[5] This is a very good little problem field for students in grade six or higher grade practicing the computation of fractions.
[6] Because 9/8 • 9/8 = 81:64
[7] In the ancient Greek mathematics (besides the so-called multiple ratios of the form n:1) the ratios of the form (n+1)**:**n with natural numbers n have to be seen as special ratios.
The investigation of different ratios of the form n:1, (n+1):n ,(n+2):n, (n+3):n and its connections, sums, differences, products and quotients as well as their presentation as musical intervals can build a little problem field too.

Prime (basis) → ***1 : 1***

Reduced Tone (Reduced Second) → ***10 : 9***

Tone (normal Second) → ***9 : 8***

Third → ***5 : 4***

Quart → ***4 : 3***

Quint → ***3 : 2***

Sixth → ***5 : 3***

Seventh → ***15 : 8***

Octave → ***2 : 1***.

As difference between one interval and the next one we now have the whole Tone with ration 9:8 and the reduced whole Tone with ratio 10:9 as well as the (diatonic) ***Half-Ton***e with ratio ***16 : 15*** which still is a "good" ratio and arises as difference between Third and Quart ([4:3] **:** [5:4]) as well as between Octave and Seventh ([2:1] **:** [15:8]).

As difference between one interval and any other one within an octave we get besides the seven normal diatonic intervals (named above) some more new Intervals (respectively ratios)[8]. Here we only want to mention the so-called ***Little-Third*** with ratio ***6 : 5***. It is defined as addition of Tone and Half-Tone (or difference of Octave and Sixth) because (9 : 8) · (16 : 15) = (2 : 1) : (5 : 3) = 6 : 5. This ratio today is very important in the minor tune (Moll-tune)[9].

[8] Finding out all such different intervals is another very good little problem field.

[9] In the ancient Greek theory of music besides the normal scale there were used scales which did not start with the basic tone (Prime) but with any other tone within the heptatonic scale (by using the following tones of the heptatonic scale as well as Octave-transpositions of them) so that the sequence of steps is instead of "Tone+Tone+Half-Tone + Tone + Tone+Tone+Half-Tone" is a different one. Today we only use the sequence of steps called minor tune with the following sequence "Tone+Half-Tone+Tone + Tone + Half-Tone+Tone+Tone".

You properly can imagine that working in this problem field there can be found a lot of more questions which concern multiplications, divisions and powers of fractions. Knowing that the ratios of the frequencies of special tones[10] correspond with the above ratios of the intervals we also can compute the frequencies of all tones in a special range.

In the Middle Ages (about 1000 AD) when organs first were built and the keys of them had to be fixed in respect to exact frequencies the seven steps of an Octave were fixed stating with the Tone named C. But because also scales starting not only with C were used between the fundamental keys there were installed the so-called black keys[11], five within one Octave so that the Octave was nearly cut into twelve Half-tones. With this five different scales nearly tuned in the diatonic way music could be played in five different heights. But all other scales did have a bad tuning.

So the question came out to divide the octave exactly into twelve equal intervals. This problem was solved first towards 1700 but it became well-know not until the middle of the 18th century when in mathematics the handling with irrational roots became usual because the "Division" of an Octave into twelve equal parts corresponds to the 12th root of 2. The scale basing on this interval is called a scale with "**well-tempered**" tuning. For it we have the following intervals

Prime $\rightarrow 1$

Second $\rightarrow (\sqrt[12]{2})^2 = \sqrt[6]{2} \approx 1{,}1225$ [*Compare*: Pyth./diatonic Second = 9:8 = 1,1250]

[10] In the end of the 19th century the frequency of the tones have been fixed so that the so-called chamber-tone "a" has the frequency of 440 Hertz.

[11] E.g. if you start with „f" then you have a normal Second to "g" and a reduced Second to "a". The next step then is not a Half-Tone but a whole Tone. So we have to install a new (black) key between these two (white) keys. Today this new key is called "b flat" (in English) or just "b" (in German whereat the following white key has the name "h").

Third → $(\sqrt[12]{2})^4 = \sqrt[3]{2} \approx 1{,}2599$ [Pyth. Third ≈ 1,2656, diat. Third = 5:4 = 1,2500]

Quart → $(\sqrt[12]{2})^5 \approx 1{,}3348$ [Pyth./diatonic Quart = 4:3 ≈ 1,3333]

Quint → $(\sqrt[12]{2})^7 \approx 1{,}4983$ [Pyth./diatonic Quint = 3:2 = 1,5000]

Sixth → $(\sqrt[12]{2})^9 \approx 1{,}6818$ [Pyth. Sixth = 1,6875, diat. Sixth = 5:3 ≈ 1,6667]

Seventh → $(\sqrt[12]{2})^{11} \approx 1{,}8877$ [Pyth. Seventh ≈ 1,8984, diat. Seventh = 1,8750]

Octave → 2.

As far as I know J. S. Bach was the first composer who used the possibility of walking through different scales in his work named "well-tempered piano". With it he spread out the interest about the well-tempered tuning among the musicians.

The new possibility of "walking" from one scale to any other one of the twelve possible scales then extensive was used by composers of the Romantic in the 19th century and in the beginning of the 20th century. Finally in the 1920th Arnold Schönberg developed the so-called "**twelve-tone-technique**" in which the basis of composing is a row of all twelve Half-Tones of an Octave.

Let me end with the hint that since the 19th century in musical theory for the notation of the interval-numbers it is used a logarithmic scale with the so-called unit "Cent". Because (as we have seen) the addition of intervals corresponds to the multiplication of the interval-numbers in this way the "addition" of intervals corresponds to the addition of the logarithms. And having not to small numbers the logarithm of an interval-number was multiplied with 1200 so that the "Cent"-value of a well-tempered Half-Tone equals to 100 (therefore the name "Cent"). Working with this scale we get a lot of opportunities for deepening the logarithm as well as musical theory.

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Sequenced problems in teaching mathematics

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Abstract: *As we are trying to find an effective way of creating sequenced problems for developing students' problem-solving skills, we examine given problem sequences. In this paper we present some results of these examinations through the example, below. We have examined the connections between the problems and the changes of cognitive aims and strategies within the sequence. With the help of these examinations we formulate a hypothesis for general requirements of the sequenced problems and we raise a few questions for further investigation.*

About sequenced problems

As Schoenfeld says: "Carefully sequenced problems can introduce students to new subject matter, and provide a context for discussions of subject matter techniques ... problem solving is not usually seen as a goal in itself, but solving problems is seen as facilitating the achievement of other goals." [8, p. 12.] In the application of problem solving in teaching mathematics we often use some kind of series of problems instead of lonely tasks. In these series of problems, there are close connection between the problems, but the kind of connections can be very different, even in a certain group. There can be the same problems with changed conditions, like in *problem fields* [4], or a group can be formed by *analogous problems* [3].

Very often a series of problems is built up in a way that the solutions of the problems should help to solve the other problems; these groups of problems are called *sequenced problems.*

Sequenced problems can be used for building up mathematical concepts, theorems and proofs, but now we examine *sequenced problems as suitable instruments for increasing the weak students' problem solving achievement.* If a math teacher would like (would like helyett elég liked) to improve the students' problem-solving skills in this way, then she/he must be able to create *effective problem sequences* by using those tasks given in the problem books. Therefore, giving some general requirements of effective sequenced problems is an important task for the researchers of didactics of mathematics.

Of course, effectiveness of a way of teaching is always a subjective thing, but there are *some necessary conditions of being powerful*, that we try to demonstrate through an example. In addition we suggest a *method, which can be used to check if a certain sequence of problems fulfils these conditions or not.*

The problems in the following example are given in the original order as they were taken from a problem book [2, p.3. problem 2073- 2077]. The sequenced problems should be given to the students in a *fixed order*, which is determined by the connections between the problems, and the changes of the cognitive aims and strategies.

The connections between the problems

There are more possible goals of using the given problem-sequence. For example, they can *make the students be familiar with vectors* or *represent a* *possible way of thinking* in proving the equation in Problem 4. An implicit, but very important goal can be *to stress using previous results in solving the problems*. The first three problems are those, where the students can practise basic operations with vectors, but these operations are necessary to get the solutions of the other problems, so by giving these tasks to students teachers can reach more than one goals at the same time.

By having a closer look at the problems and examining them according to Bloom's cognitive levels (knowledge, understanding, application, analysis, synthesis, evaluation), we can observe the connections between the problems. In the next part we summarize the concrete cognitive activities for the certain problems:

Problem 1.(2073)Let there a regular hexagon be given with its side and diagonal vectors. Find vectors with the same length, but different direction and vectors with the same direction but different length!

Knowledge: features of the regular hexagon, the concept of the vector (length, direction).

Understanding: groups have to be created from the vectors in the hexagon from two different aspects.

Application: features of the regular hexagon (equal segments, known angles).

Analysis: examining and comparing the vectors in the hexagon from the aspects of length and direction.

Synthesis: making groups of the vectors, and examining the groups.

Evaluation: examining the differences in the certain groups (if the direction is the same, what is the connection between the lengths and vice versa), looking for those vectors, which are in the same group from both of the aspects (equal vectors).

Problem 2.(2074)a. There are ***a,*** *and* ***b*** *vectors pointing from the midpoint of a regular hexagon to two neighbour vertexes of the hexagon. Express the side and diagonal vectors of the hexagon with* ***a,*** *and* ***b*** *vectors!*

b. There are ***a****, and* ***b*** *vectors pointing from a vertex of a regular hexagon to the neighbour vertexes. Express the side and diagonal vectors of the hexagon with* ***a****, and* ***b*** *vectors!*

Knowledge: features of the regular hexagon, vector-operations (addition and subtraction using the parallelogram-method).

Understanding: the side and diagonal vectors can be expressed with the given vectors.

Application: results of Problem 1. (equal vectors, 1:2 ratio of certain lengths), parallelogram-method.

Analysis: comparing the certain vectors to the given vectors, looking for those vectors that can be expressed easily.

Synthesis: comparing all of the vectors to those that are already expressed.

Evaluation: checking whether we have expressed all vector, examining the vectors according to Problem 1.

Problem 3.(2075) In the regular hexagon ABCDEF there are ***a***= $\overrightarrow{AB}$*, and* ***b***= $\overrightarrow{AF}$ *vectors given. Let the midpoints of the sides CD and EF be P and Q. Express* $\overrightarrow{AD}$, $\overrightarrow{AC}$ and $\overrightarrow{PQ}$ *vectors with* ***a,*** *and* ***b*** *vectors!*

Knowledge: vector-operations (addition, subtraction, multiplication with a scalar), vector pointing to the midpoint of a segment, vector pointing from a given point to another, results of Problem 2.b.

Understanding: $\overrightarrow{AD}$, $\overrightarrow{AC}$ and $\overrightarrow{PQ}$ vectors can be expressed with the given vectors (like in Problem 2.b.).

Application: results of Problem 2.b, vector pointing to the midpoint of a segment.

Analysis: expressing $\overrightarrow{AP}, \overrightarrow{AQ}$ vectors in triangles ACD and AEF.

Synthesis: expressing $\overrightarrow{PQ}$ with $\overrightarrow{AP}$ and $\overrightarrow{AQ}$.

Evaluation: by comparing the result with the results of Problem 2.b, another method can be realized, because we can see that $\overrightarrow{PQ} = \frac{3}{4}\overrightarrow{CF}$.

Problem 4.(2076) Draw the regular hexagon ABCDEF and show that:

$$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AE} + \overrightarrow{AF} = 2\overrightarrow{AD}\,!$$

Knowledge: vector-operations, concept of equal vectors, features of the regular hexagon, the results of Problem 2.

Understanding: the vectors are equal if their length and direction is the same or if they can be expressed with other vectors in the same way.

Application: substituting the results of Problem 2.

Analysis: expressing the certain vectors with **a** and **b** vectors**.**

Synthesis: comparing the two sides of the equality.

Evaluation: by examining the $\overrightarrow{AB}+\overrightarrow{AF}$ and $\overrightarrow{AC}+\overrightarrow{AE}$ sums we can find an other way of thinking

Problem 5.(2077)a. Draw the regular hexagon ABCDEF and its midpoint, O! Determine the sum of $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OE}$ *vectors!*

Knowledge: vector-operations, the results of Problem 2, the concept of zero vector.

Understanding: we are looking for a vector, which is equal to the sum of the given vectors.

Application: we can use the results of Problem 2.

Analysis: expressing the certain vectors.

Synthesis: examining the sum of the vectors.

Evaluation: we can recognize the equilateral triangle ACE, so the problem can be observed from other points of view.

Problem 5.(2077)b. Draw the ABCDEF regular hexagon and its midpoint, O! Determine the difference of $\overrightarrow{AB}$, $\overrightarrow{ED}$ *vectors!*

Knowledge: vector-operations, the results of Problem 1, the concept of zero vector.

Understanding: we are looking for a vector that is equal to the difference of the given vectors.

Application: we can use the results of Problem 1.

Analysis: examining $\overrightarrow{AB}$ and $\overrightarrow{ED}$ vectors according to Problem 1.

Synthesis: recognizing that $\overrightarrow{AB}$ and $\overrightarrow{ED}$ vectors are in the same class from both of the aspects of Problem 1. (equal vectors).

Evaluation: we can find more pairs of vectors for which the difference is zero vector.

Problem 5.(2077)c. Draw the ABCDEF regular hexagon and its midpoint, O! Find four vectors in the figure for which the sum of the vectors is zero vector!

Knowledge: vector-operations, the results of Problem 1, the concept of zero vector.

Understanding: we are looking for vectors, for which the sum is zero vector, as in Problem 5.a.

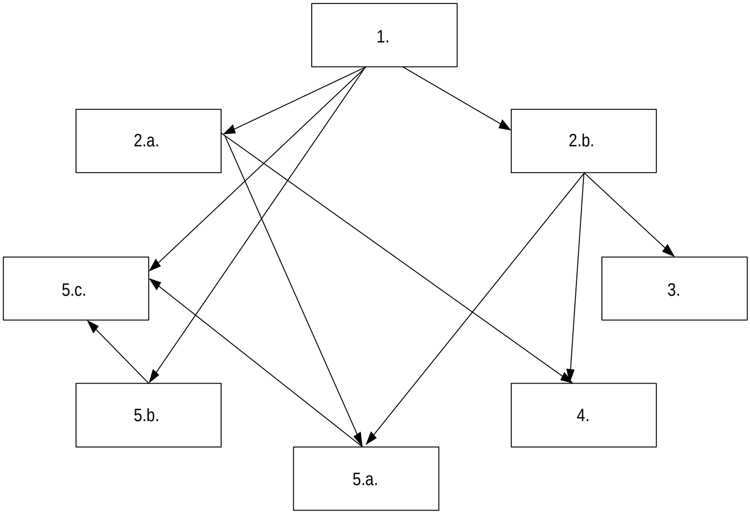
Application: we can use the results of Problem 1, and Problem 5.b.

Analysis: finding pairs of vectors, like in Problem 5.b.

Synthesis: finding rectangles in the figure, like rectangle ABDE.

Evaluation: comparing more of the solutions.

If one has analyzed the sequenced problems in this way, then it is well-observable, that the solution of a problem can be used for solving the next problems. After we have solved Problem 2, then it is easy to solve Problem 4. and Problem 5.a, by only substituting the right expressions of the given vectors. By solving Problem 1, we examine all of the vectors in the hexagon so we can find equal vectors, even if it is not the task, and this implicit help makes the next problems to be easier. In the next oriented graph we have represented these connections.



By examining the number of the edges in the graph, we can state that Problem 1 has the most edges starting from the problem (we can state that there are the most edges starting from Problem 1). This is *the basic problem of the sequence*. The solution of this problem helps solve most of the other problems, so this problem has to be easy to solve even for the weak students. With help of the graph of connections we can discover *critical problems;* if we would left them, the connected graph would disintegrate into parts. In the given example this is Problem 2.b. We have to be aware of those sequenced problems, which contain critical problems, because if the students are failed with them, then they probably will not be able to solve the others (Problem 3.). Usually, the selected problems can be solved in other ways, without help of the previous solutions, but the weak students often give up work in the situations we have mentioned. If there are no critical problems in the sequence, then students can go further even if they had failed to solve a certain problem. A required condition of not to have these kind of problems in the sequence is, that *not only the neighbour problems should be connected!*

We have chosen this example, because it is right for presenting the mentioned conditions for effective sequenced problems:

- Not only the neighbour problems should be connected.
- The basic problem(s) should be selected carefully.
- There should not be critical problems in the sequence.

By examining the problems according to Bruner's cognitive levels we can make the graph of connections for the sequence and by observing the graph we can decide if a sequence of problems fulfils the conditions or not.

Open questions

In the analysis of the given problem-sequence we focused on the connections between the problems from the aspect of developing the weak students' problem solving skills. What about those students, who are talented in mathematics? The mentioned example contains such problems that can be solved without using the solutions of the previous problems in the sequence. Although the most obvious problem solving method in this sequence is reducing, I am sure, that in a classroom there would be more ways of thinking. In Problem 3. for example, $\overrightarrow{PQ}$ vector can be determined by using the side vectors, we have expressed before, or one can say, that $\overrightarrow{PQ}$ vector is the arithmetical average of side vector $\overrightarrow{DE}$ and diagonal vector $\overrightarrow{CF}$. Problem 4. and Problem 5. can also be solved without using the previous results, for example, one can say for Problem 5.a, that $\overrightarrow{OA}$, $\overrightarrow{OC}$ and $\overrightarrow{OE}$ vectors are the rotated side vectors of triangle ACE multiplied by 2/3, so their sum is 0. As we could see before, one can get to the mentioned alternate solution for Problem 3. by looking back to the results of Problem 2.b, but there is no such help for the alternate solution of Problem 5.a. in the previous tasks. A disadvantage of using sequenced problems is that the talented students can be hindered in divergent thinking by the striking connections between the problems.

Another important question is about the force of these connections. How could we know whether the problems, that we have found connected, would be also connected in the students' mind? How could we classify these connections? I think if we could find the answers of these questions, then the conditions of powerful sequenced problems will be more clarified.

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What does it mean to be sensitive for the complexity of (problem oriented) teaching?

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Abstract: Teaching is complex. And often requested problem oriented mathematics instruction is even more complex than traditional instruction. From that special demands on the teacher and because of that on teacher education arise. At least teacher education at university should include attempts to sensitize becoming teachers for the complexity of (problem oriented) mathematics instruction first. For development and testing of new complementing elements of education, there has to be constructed a diagnostic instrument and to be elaborated more precisely, what is meant by sensitivity for complexity. First results will be given in this article.

Theoretical Framework

"Teaching is acting and deciding in a complex system." This is said very often, it is written very often in popular as well as in specialized educational, psychological and didactic literature.[1] But what does it actually mean? Whereby problem oriented mathematics instruction (POMI) is more complex than traditional instruction? First I want to approach possible answers by a paradigmatic example: the use of a fascinating folding paper-problem in a math lesson.

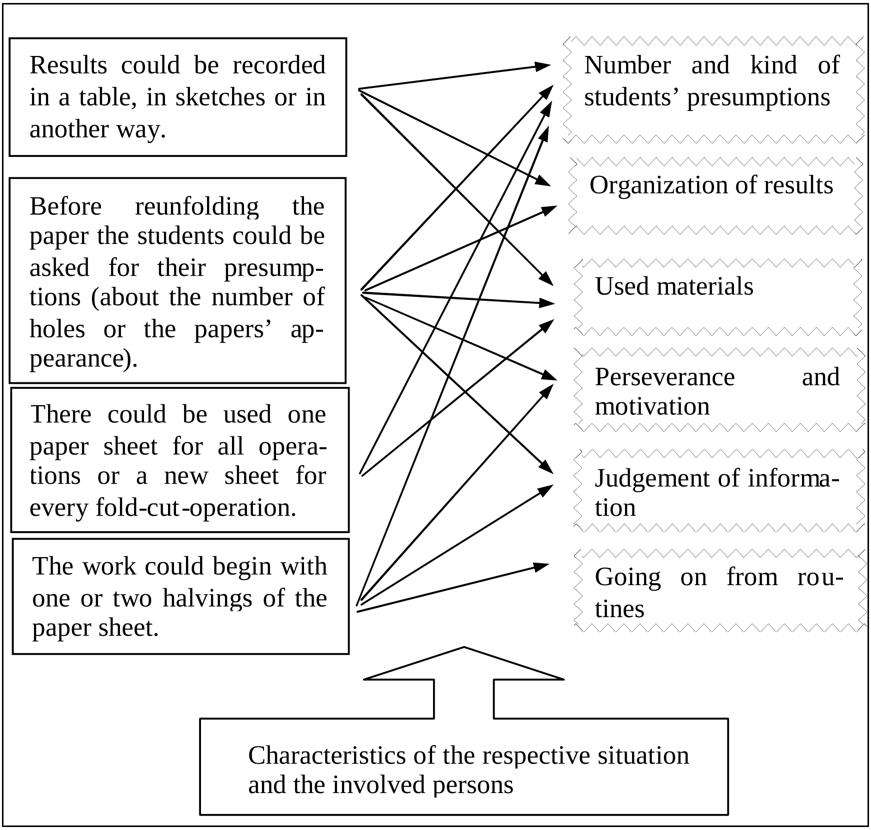
[1] For references see e.g. Fritzlar (2004a).

The *Faltproblem* (folding paper-problem): A sheet of usual rectangular typing paper is halved by folding it parallel to the shorter edge. The resulting double sheet can be halved again by folding parallel to the shorter edge and so on.
After n foldings the corners of the resulting stack of paper sheets are cut off. By opening the paper, it will be seen that (for $n>1$) a mat with holes has arised.
Find out and explain a connection between the number n of fold-cut-operations and the number $A(n)$ of holes.[2]

In almost 50 lessons in different grades and school types the *Faltproblem* showed its special potentials for problem orientation, but special characteristics of POMI and resulting demands on the teacher have appeared too. Here I can outline only one situation, which can occur during a lesson about the *Faltproblem*:[3]

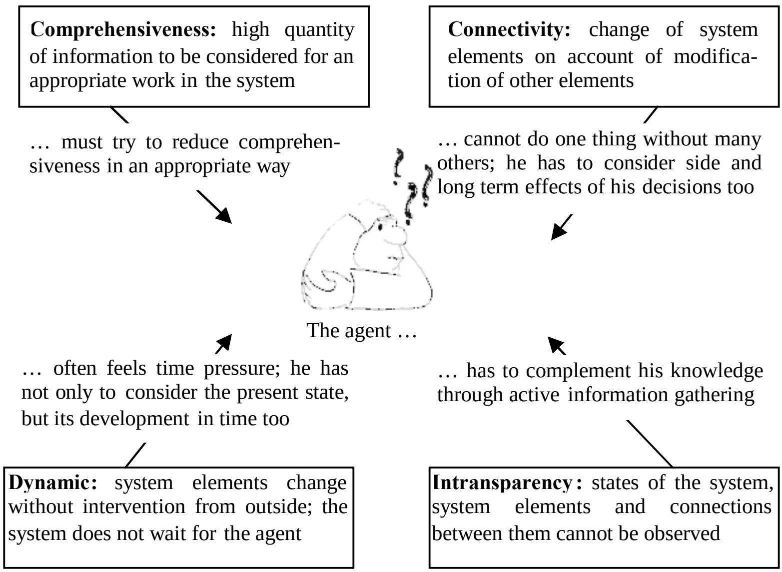
As an introduction into the problem the teacher possibly wants to work on some fold-cut-operations together with his students. These first steps can be varied in many details. The figure shows some possibilities for variations and probably influenced characteristics of students' problem solving processes. The arrows can only hint at the multitude of connections, additionally influenced by special characteristics of the situation and involved persons.

2 This problem was developed by KARL KIEßWETTER (e.g. KIEßWETTER & NOLTE 1996) to use in an entrance examination of the University of Hamburg.
3 More examples can be found in FRITZLAR (2004c).



I hope this simplifying figure stimulates to further considerations, so that the reader realizes the richness of only this individual situation and the many connections between conditions of and decisions during the lesson, features of the lesson course, side and long term effects. Additionally in such situations the teacher usually has to decide under time pressure. And it is impossible for him (maybe it is impossible in general) to know all influencing factors and to consider them in an appropriate way.

Generally teaching can be characterized by the following features and resulting demands which are typical for complex systems:[4]



There is absolutely no doubt that teaching is complex in this sense, but beyond this POMI is characterized by an additional high complexity concerning math-cognitive aspects: During a lesson many al least partly different problem solving processes of students appear simultaneously and go very quickly, which should be watched and supported by the teacher if necessary. These processes are influenced by numerous anthropogenic and socio-cultural conditions and especially by (sometimes inconspicuous) teaching-decisions in many ways. So unexpected matters occur very often and the teacher must react in only few seconds. On these occasions he has possibly to free himself of or at least question own views.

[4] Therefore they will be described for any complex system in general.

And POMI is very low transparent because the teacher cannot look into pupils' minds, that's why their problem solving processes are often difficult to understand. In addition the teacher is no longer the only (and authoritarian) source of information and he has to give his students more scope for doing mathematics (FRITZLAR 2004b).

Conclusions and goals of research

From the described complexity of POMI on the one hand and the goal of a stronger problem orientation of math education on the other hand, special demands on teacher education arise, which are not fulfilled up to now. Also the first part of teacher education at university ought to be complemented in an appropriate way. At least we should try to *sensitize* teacher students for the complexity of POMI. In this context I designate – as a preliminary approach – a teacher, a teacher student or more generally an agent as *sensitive*, if he is *aware of the complexity* of POMI, *of special demands* arising from it and *of limits of his possibilities to decide and to act* in an appropriate way. (A more detailed operationalization is only possible by consideration of the respective context.)
But for this purpose of sensitization there has to be done basic works first. In particular a diagnostic environment has to be created, which enables us to get information about the teacher students' initial situation and to evaluate possible complementing elements of education. But how and in which situations can you notice sensitivity for complexity? How can an appropriate diagnostic instrument look like?

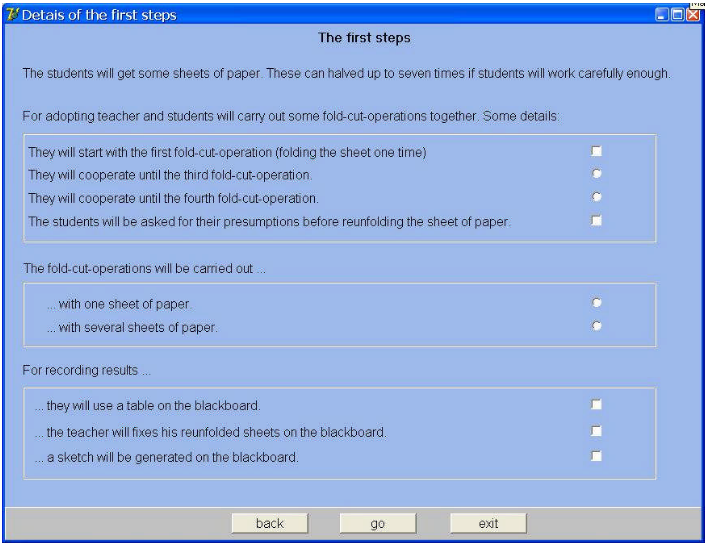
Research work

It seems to be clear that sensitivity for complexity appears above all in analyzing and evaluating of decision situations connected to POMI. First of all you might think of real situations as most suitable for obtaining hints for an agent's degree of sensitivity, but these would cause some disadvantages, e.g.:

- It seems important to me to take into account, that sensitivity does not mean successfully coping with complex situations. It rather means an appropriate subjective modelling which could be hardly scrutinized within real situations.
- Real situations are unrepeatable and hardly to vary systematically.
- In real situations the agent normally has to act under time pressure and his decisions are also influenced by his ability to realize these decisions ("pressure of performance").

That's why I decided to develop an artificial diagnostic instrument consisting of two parts.

First part: an interactive computer scenario. I analyzed the videotaped lessons about the *Faltproblem* particularly in regard to features of students' problem solving processes and connections between them, conditions of the lesson and teaching decisions during the lesson. The data were joined to a realistic interactive scenario which enables a user to work on the modelled network. In the role of the teacher the user can work on special teaching situations and for instance test different decisions or same decisions under different conditions; upon the computer program reacts and brings the next decision situation to him. At the end the user gets an assessment of his decisions in regard to his own goals for the lesson. The next figure shows the realization of the decision situation described above and the main structure of the scenario.



selection of a class

statistical data, students' abilities, experiences concerning problem solving, collaborative working, ...

↓

selection / prioritization of lesson goals

short-term goals (regarding the *Faltproblem*),
long-term goals (regarding heuristic competences, attitudes, ...)

↓

teaching decisions

presentation of the problem, questions and instructions to the students, used materials and media, ...
Given alternatives can be selected or supplemented.

reactions by computer regarding lesson situations, students' problem solving processes and results

students' activities, presumptions, argumentations, (approximate) assessment concerning students' motivation, involvement, …

↓

assessment of teaching decisions

comparison between achieved and planned goals, assessments concerning consistence of decisions, extent of control by the teacher, comprehensiveness of mathematical doing, ... →

I see at least the following potentials of the computer scenario:

- A scenario allows a satisfactory complex modeling with concentrating on very important but often more or less ignored math-cognitive aspects. Through the program's interactivity a complex network of decision situations arise for the user, which is comparable to real teaching.
- Unlike reality decision situations can be explored repeatedly (as often as the user want) and without time pressure. In addition it is unimportant for the user, if he is able to execute his plans.[5] Altogether a scenario can model decision situations realistically, and it enables ways to analyze these situations, which do not exist in reality but whose use can provide some hints for the degree of sensitivity.
- Modelled situations can be varied systematically. By this the user can experience complexity of teaching in a special way and the teacher educator can analyze his examination of this complexity.
- As many students as wanted can work with the scenario, and it can be handled in an easy way.

[5] This could be important in particular for teaching novices.

Second part: an interview. I developed a special interview concerning the use of the *Faltproblem* in a math lesson. In the course of the interview the subject works on few situations, in which he has to decide about the further lesson course, he has to judge suggested decisions or to interpret problem solving processes and results of students. With this interview further relevant information about the degree of sensitivity are to be obtained by compensating observed disadvantages of the computer scenario. In particular I see the following complementing potentials of the interview in this regard:

- The subject will be explicitly asked to develop (appropriate) variants and alternatives. (In the scenario the user can also give additional alternatives, but in my empirical studies this feature was hardly used.)
- Considerations of the subject (interpretations of described situations, reasons and goals of decisions, …) can be recorded in detail. (In the scenario the user is also asked to verbalize, but the extent of comments varied very much in my investigations. By the multitude of situations to be worked on this is just a matter of energy and stress too.)

The following figure shows the realization of the previously described entrance situation in the interview:[6]

In a math teaching experiment the *Faltproblem* is supposed to be dealt with in the <u>class 5a</u>.[7] `How could the first steps of dealing with the problem be engineered? Give some different variants!`

[6] Questions and requests on the subject are type written.
[7] Information about the class is offered as in the scenario.

A teacher student has planned first steps as following: The pupils and the teacher work together on the first fold-cut-operations. After folding and cutting the paper the pupils are always asked for their presumptions (before unfolding the sheet of paper).

How do you judge this decision of the teacher student?

How could details of this start in dealing with the *Faltproblem* be designed and varied? Give some different variants!

Among other things the following variants are possible:

- Before unfolding the paper the pupils are asked for their presumptions regarding to the *number* of holes *or* regarding to the *appearance* of the unfolded sheet of paper.
- The work on the *Faltproblem* starts with the *first or* with the *second folding*.
- The work on the *Faltproblem* starts with the *first or* with the *second folding*.
- All fold-cut-operations are carried out at *one sheet of paper or* the pupils get another *sheet of paper for every fold-cut-operation*.

Please name some possible *effects* and possible *advantages* and *disadvantages* of these variants!

How would *you* engineer the first steps in the class 5a? Please, give some reasons for your decisions!

Results and discussion

The diagnostic instrument was applied in a first empirical study with twenty teacher students from five universities in Germany.[8] Up to now I was able to analyze data of subjects' working on the computer scenario in detail.

But what does it mean to be sensitive for the complexity of POMI in this special situation? Since this field of research is new, worldwide there hardly exist experiences about it and so my investigation can "only" be a pilot study, it seems appropriate to derive criteria for the degree of sensitivity not only from theoretical considerations but from gained data too.

Certainly sensitivity for complexity (sfc) is a complex quantity itself which cannot be characterized by an only number. Grounded on gained data about the working on the computer scenario I developed a four-dimensional vector, which describes the user's degree of sensitivity. The following table shows a short description of these components and, in italics, main results of my pilot study; for more details I have to refer to FRITZLAR (2004a):

Exploratory behavior:	
representing quantitative and qualitative aspects of the user's exploration of the scenario.[9]	*In general students explored the scenario only to a small extent; possibilities for systematic testing of teaching-decisions also on different conditions were hardly used.*
Context sensitivity:	

[8] I want to thank for their support Prof. Dr. Regina Möller (University of Erfurt), Prof. Dr. Marianne Nolte (University of Hamburg), Prof. Dr. Günter Graumann (University of Bielefeld), Prof. Dr. Friedhelm Käpnick (now University of Münster) and Prof. Dr. Bernd Zimmermann (University of Jena).

[9] e.g. number of loops and jumps back within the program and number of different modes of representation the problem

representing to what extent the user referred in decision-situations to problem solving processes of students, aspects of the mathematical content, or more social aspects (motivation, teaching methods, …) of the lesson.	*Many users realized that the scenario focuses on math-cognitive aspects of POMI. Anyway features of students' problem solving processes were considered only superficially and to a very small extent.*
Inconsistence:	
representing the percentage of the user's decisions, which are interpreted to be not consistent with modelled aspects of the lesson (in particular with features of students' problem solving processes).	*This component varied very much. Because of the small extent of exploration it was not always possible to obtain a result.*
Reflectivity:	
representing the extent of (critical) reflection (e.g. of quality of modelling by a computer program, of own decisions and decision behavior, …).	*Reflectivity was low in general. Users hardly reflected on connections between conditions of and decisions during the lesson and students' problem solving processes.[10] Multidimensionality of decisions was rarely taken into account; meta-cognition was hardly perceptible.*
There are no objections against the independence of the components of the sfc-vector.	
I could not find specific sensitivity types in the experimental group.	

[10] Also from there arose only few motives for exploration.

The analysis of data obtained by interviewing the subjects (second part of the diagnostic instrument) could not be completed yet. Theoretical considerations e.g. about the psychological approach of "cognitive complexity" (MANDL & HUBER 1978) and first impressions from obtained data indicate, that in particular the structure of argumentation might be an important criterion for sfc. Among other things in this connection you can distinguish differentiated vs. absolute judgement and conditional vs. absolute deciding of the subject.

I hope I am able to present more results in a few months and proceeding from them to characterize the degree of sensitivity for complexity still more precisely.

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Anatomy of a contest problem

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Abstract: *In this article we will be concerned with the heuristics of solving a mathematical contest problem. We will discuss a problem of the XIV-th International Hungarian Mathematical Contest (Miskolc, Hungary, 2005). We are analyzing the written solutions of Problem 5 (grade 9) based on the works of the 59 contestants. The main aims of our investigations were: to identify the applied solutions, the strategies, the misconceptions of the contestants and to draw some conclusions for the fostering of talented students.*

The XIV-th International Hungarian Mathematical Contest

This year in Hungary there was organized the XIV-th International Hungarian Mathematical Contest for grades 9-12. Each grade has its own problem series, which consists of 6 problems. The contestants get 10 points for the first solution of a problem and 2 points more for the second solution or for a generalization. The minimum of points is 1 point.

The fundamental goals of these competitions are to awaken interest in mathematics and to develop talents. A contest problem illustrates in a miniature the process of creative mathematics. A good problem tells us a story of mathematical creativity and beliefs. Therefore it is very important to deal with the mathematical contest problems and analyze the pupils' heuristically solutions, their strategies or tactics and the ways of their thinking. We can use these experiences in teaching of talented pupils for the next contests.

The other reason for examining the works of the competitors is to improve posing and formulating the problems of the competitions in the future. The proposed problems are always hard and require mathematical knowledge, experience and mathematical skill, sharp and quick thinking and good computing competence.

We, I and a young teacher, corrected and evaluated the solution of Problem 5 given to pupils of grade 9. We chose a geometrical problem, a problem of Euclidean plane-geometry. We knew that geometry is hard for the best pupils too. Why? I think that it needs more heuristics and using both halves of our brains. Mathematical knowledge is not enough. The pupils are not accustomed to heuristical methods, and many of them do not like to sketch figures. It is important to learn more about strategies for problem solving, to sketch methods for the competitors. During the competition the pupils were many times within an ace of solving the problem, but they did not see and could not find the missing step.

Problem 5 (grade 9)

Let ABCD be a trapezium. The length of the diagonals AC, resp. BD is 9 and 12. These diagonals are perpendicular to each other and $|AB|\cdot|CD| = 50$*.*
Consider the value of the sum: $|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2$.

Problem 5 as a geometrical problem was based on well-known school-material. It was complex and its solution was like a labyrinth. One had to go step by step, but there were a lot of similar and different steps. It was not easy to find the right way. We could come to an impasse. It is not easy to find the best solution of a problem, or to generalize it during the competition. The best solutions come often later, after the competition, when there is a time to discuss the problem among the competitors and tutors. This is a good method for preparing the talented pupils to the next competitions.

Results

We corrected the works of 59 pupils: Hungary (21), Romania (15), Serbia (8), Slovakia (13), Ukraine (2). 7 pupils (11,86%) gave correct solutions, 5 pupils (8,47%) gave almost correct solutions, 3 pupils (6,78%), tried to understand the problem and began solving it, they had partly result, 44 pupils (75,8%) had no result, they made some starting steps, drew figures, made notations, and wrote the data. Only the best pupils of grade 9 were successful in solving Problem 5, or we can formulate our statement inversely too: those pupils were the winners who could solve the Problem 5, so it became a dividing line. (Fig.1)

	1 point	2 points	3 points	4 points	5 points	6 points	7 points	8 points	9 points	10 points	10 +2 points
Hun	4	5	3	1	-	-	-	2	1	5	-
Ro	4	4	4	1	-	-	-	-	-	1	1
Sr	2	1	2	-	-	1	1	-	1	-	-
Sl	6	3	4	-	-	-	-	-	-	-	-
Ukr	1	1	-	-	-	-	-	-	-	-	-
Total	**17** 28,8%	**14** 23,7%	**13** 22%	**2** 3,4 %	**0** 0%	**1** 1, 7 %	**1** 1,7 %	**2** 3,4 %	**2** 3,4%	**6** 10,2%	**1** 1,7 %

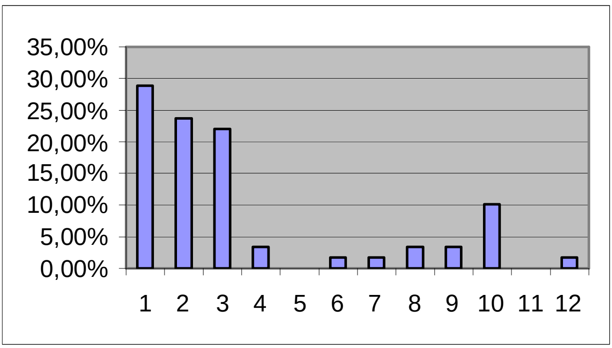


Fig.1

How are thinking the contestants?

If we look at the work of the contestants in solving Problem 5, we can see typical characteristics. The decisive step in the solution of geometrical problems may be to make a figure and to introduce appropriate auxiliary lines. It was very important when the pupils recognized some familiar feature in the given geometrical figure (trapezium, right angled triangles), or in computing they may recognize a complete square. There are well-known theorems for the trapezium (midline, area, similarity), for right angled triangles (Pythagorean Theorem, height theorem). When they want to apply the Pythagorean Theorem for the perpendicular diagonals they have to translate one of them to get a right angled triangle. It was possible to apply the Pythagorean Theorem to four right angled triangles. Some of them noticed that certain lines formed a pair of similar triangles.

In this paper we will be concerned with the heuristics of solving a mathematical contest problem, namely the Problem 5, based on 59 students' work. We shall discuss different problem solving strategies and analyze their typical tactics, results, beliefs and missing steps.

Analysing the solutions from the point of view of applied strategies

1. The use of general theorems

Problem 5 was a special case of Euler's theorem.

Euler's theorem

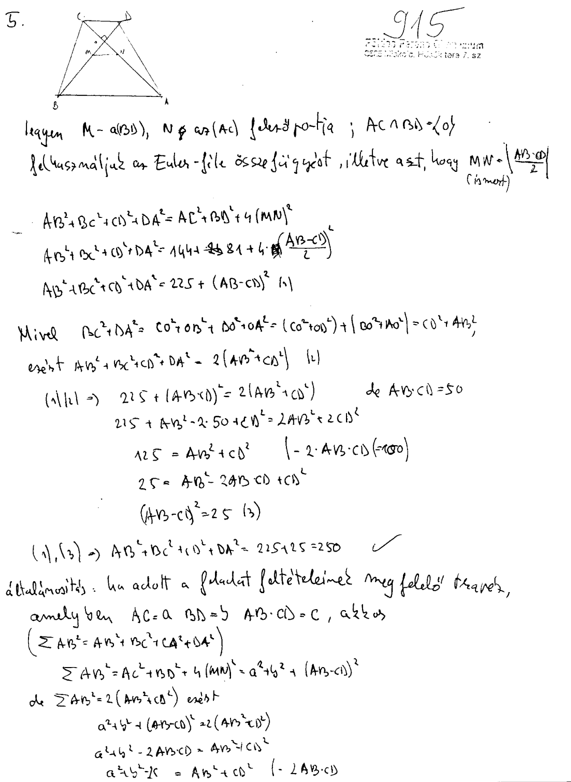
If M, resp. N are midpoints of the diagonals AC, resp. BD of a quadrilateral ABCD, then $|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 = |AC|^2 + |BD|^2 + 4\,|MN|^2$.

In our case quadrilateral *ABCD* was a trapezium, so $|MN| = \frac{1}{2}\big|\,|AB| - |DC|\,\big|$.

In Hungary the Euler's theorem is not a well-known theorem, and its knowledge is not expected even from the best pupils.

Cases

- Contestant 915 (he was the winner) knew Euler's theorem and applied it in the given special case. He gave a generalization too. The generalization is easy; it goes in the same way as the concrete case. (Fig.2.)
- The problem poser proved directly the special case of Euler's theorem.



5\.

915

legyen M - a(BD), N az (AC) felezőpontja ; $AC \cap BD = \{O\}$

felhasználjuk az Euler-féle összefüggést, illetve azt, hogy $MN = \left|\frac{AB-CD}{2}\right|$ (ismert)

$AB^2+BC^2+CD^2+DA^2 = AC^2+BD^2+4(MN)^2$

$AB^2+BC^2+CD^2+DA^2 = 144+81+4\cdot\left(\frac{AB-CD}{2}\right)^2$

$AB^2+BC^2+CD^2+DA^2 = 225+(AB-CD)^2$ (1)

Mivel $BC^2+DA^2 = CO^2+OB^2+DO^2+OA^2 = (CO^2+OD^2)+(BO^2+AO^2) = CD^2+AB^2$,

ezért $AB^2+BC^2+CD^2+DA^2 = 2(AB^2+CD^2)$ (2)

(1),(2) ⇒ $225+(AB-CD)^2 = 2(AB^2+CD^2)$ de $AB\cdot CD = 50$

$225+AB^2-2\cdot 50+CD^2 = 2AB^2+2CD^2$

$125 = AB^2+CD^2$ $\quad | -2\cdot AB\cdot CD (=100)$

$25 = AB^2-2AB\cdot CD+CD^2$

$(AB-CD)^2 = 25$ (3)

(1),(3) ⇒ $AB^2+BC^2+CD^2+DA^2 = 225+25 = 250$ ✓

általánosítás: ha adott a feladat feltételeinek megfelelő trapéz, amelyben $AC=a$ $BD=b$ $AB\cdot CD=c$, akkor

$(\sum AB^2 = AB^2+BC^2+CA^2+DA^2)$

$\sum AB^2 = AC^2+BD^2+4(MN)^2 = a^2+b^2+(AB-CD)^2$

de $\sum AB^2 = 2(AB^2+CD^2)$ ezért

$a^2+b^2+(AB-CD)^2 = 2(AB^2+CD^2)$

$a^2+b^2-2AB\cdot CD = AB^2+CD^2$

$a^2+b^2-2c = AB^2+CD^2$ $\quad | -2AB\cdot CD$

Fig. 2

- The knowledge of the Pythagorean Theorem was not sufficient by itself. Applying the Pythagorean Theorem was connected with other methods (symmetry, translation, area of trapezium, similarity, theorem of parallel intersecting lines, recognizing of a complete square).
- Contestant 937 gave a complete, right solution. He was applying the theorem of Thales (theorem of intersecting parallel lines) and Pythagorean Theorem. He gets that $|AO| = 4/3|BO|$, $OC = 4/3|DO|$ (O is the intersecting point of the diagonals). He squares $|AB| \cdot |CD| = 50$ and used that $|AB|^2 = |AO|^2 + |OB|^2$, $|CD|^2 = |DO|^2 + |OC|^2$. From these follows that $|BO| \cdot |OD| = 18$. He computed the value of the term $|AO| \cdot |OC| + |BO| \cdot |OD|$, applying the Pythagorean Theorem four times and he got the result.

2. Analytic geometry

I observed that on contest level some pupils choose the method of analytic geometry for solving plane-geometrical problems. In our case two students chose this method. They placed in a special situation the given figure in the Descartes coordinate-system and very elegantly solved the problem.

Cases

- Student 64 placed the trapezium in the coordinate-system so that the two diagonals were joining the x-axis and the y-axis.
- Student 78 used a false belief, he constructed a rectangular trapezium.

3. Geometrical transformations: translations and similarity

There are some common ideas and characteristic methods in classical plane geometry. If we deal with a trapezium we have to know that the translation of its sides or diagonals is a very useful tool. The above mentioned moments are part of a greater chain of thought and form the first useful step forwards the solution.

Cases

- A lot of the contestants chose for the first step of the solution the translation of the perpendicular diagonals and formed a right angled triangle, where the lengths of the legs were 9 and 12. With the help of the Pythagorean Theorem they considered the length of its hypotenuse. They got that $|AB| + |CD| = 15$. It was given that $|AB|\cdot|CD| = 50$. From the quadratic equation $x^2 - 15x + 50 = 0$ they got $|AB| = 10$, $|CD| = 5$.
- In carrying out the solution the second step of the problem solvers was different. They followed many helpful strategies:

1. Similarity of *ABM* and *DCM* triangles (*M* was the intersection point of the diagonals) and applied the Pythagorean Theorem (contestants 63,907,962)
2. They computed the area and the height of the trapezium *ABCD*, and applied the Pythagorean Theorem (contestants 927, *947,* 959).
3. They applied sometimes Pythagorean Theorem (*960*).

- The solution of student 67 is based on similarity of the *ABM* and the *DCM* triangles. He used special notations: $AM=k\cdot AC$, $CM=(1-k)\cdot AC$, $BM = k\cdot DM = (1-k)\cdot BD$, $k\in\,]0;1[$. Squaring the condition $AB\cdot CD = 50$, he got the $9k^2+9k+2 = 0$ quadratic equation with the roots $k_1= 1/3$, $k_2 =2/3$, and from this immediately the value of the requested square sum.

There were other tactics too:

- Contestant 916 has put $AM=x$, $DM=y$, so $x:(12-x)=(9-y):y$, $x=12-4/3\,y$. He solved the quadratic equation $25y^2 -225y + 450 = 0$, got $y_1=3$, $y_2=6$ and could compute the value of the requested square sum.
- There was a better notation too: $DM = x$, $CM = y$. In this case $y = 4/3\,x$. Contestants 911, 914, 925, 961, 964, 68, 78 made this choice, but they

could not end the solution. The main reason was that they made mistakes in solving of quadratic equation, or in the algebraic transformations.

4. Exploitation of Symmetry Principle

It was possible to solve Problem 5 with exploitation of Symmetry Principle. This method was the shortest. In our case $|AB|^2 + |CD|^2 = |DA|^2 + |BC|^2$, so $|AB|^2+|BC|^2+|CD|^2 + |DA|^2= 2(|AB|^2+ |CD|^2)$.

Cases

- Student 66 gave such a kind of solution. In the first step she denoted with x, $12-x$, y, $9-y$ the two parts of the diagonals, then wrote the sum of the squares $|AB|^2+|BC|^2+|CD|^2+|DA|^2$ and used the method of completing the square. Her method was general, only in the last step put she the data in the terms.
- The method of the contestant 65 was a combination of translation, algebraic method (completing a square) and symmetry principle. (Fig.3)

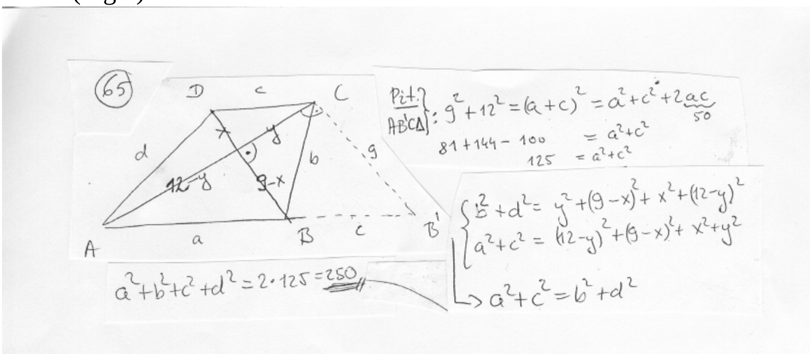


Fig. 3

- Contestant 934 was absent-minded. He had forgotten to write the given relation $|AB|\cdot|CD| = 50$. He found the algebraic symmetry, but he could not reach the solution in consequence of the missing value.

He was within an ace of solving the problem, but he did not find the missing step.

5. Working backwards and forwards

I met with a special combination of the methods backwards and forwards. This was the tunnel building method. The contestants began the proof from two directions, from backwards and forwards too. They hoped that the two tunnels will meet punctually, or there will be only a little gap in the connection of their statements, arguments. This strategy was successful.

Case

- We found the tunnel building method in the work of the contestant 923. In the middle of the proof there was a gap. He found that $|AB|^2 + |CD|^2 = 125$, but he did not see the symmetry. So he tried to compute some proportions exploiting the similarity of the triangles *ABM* and *BCM.*

6. Circular reasoning

In solving Problem 5 it was important to apply many times the Pythagorean Theorem. Without conception this method did not lead to the solution, may be that the contestants got a more complicated term, or they went back to an identity.

Cases

- Some contestants wrote the Pythagorean Theorem to the triangles *AMD, ABM, BMC, DMC* and they carried out the squaring of the terms, but they could not make a progress. The form of the term would be more complicated.
- The other tactic was to compose the sum of squares

$$|AB|^2 + |BC|^2 + |CD|^2 + |DA|^2,$$

but the contestant 910 could not find the value of $|AM|\cdot|MC|+|BM|\cdot|MD|$, therefore he tried to do something with the help of similarity and he got a more complicated term.

- Contestant 909 wrote a lot of connections, but at last got an identity.

Problems of incomplete solutions, mistakes and false trials

7. Problems of choosing notations, problems of drawing figures

With Problem 5 there was not given the figure of trapezium *ABCD.* The first problem was: How can we draw the trapezium, and how can we denote the vertices of the trapezium?

The formulation of a problem needs knowing conventions. On the International Hungarian Mathematical Competitions the mother language is common (Hungarian), but the school-system, the curricula and mathematical conventions are different.

Cases

- For two contestants (65, 67) the absence of a figure for the trapezium caused a problem. They examined two versions:

1. Usual case: $|AB| > |DC|$, $AB \| DC$.

This version was chosen by the other contestants, it is the only possible case.

2. Other case: $|AD| < |BC|$, $BC \| AD$. This case was not possible.

As we mentioned one of the first step in solving Problem 5 was to make a figure. It was not easy to draw a trapezium with perpendicular diagonals. Some contestants at first drew the diagonals and later the parallel sides.

- 6 contestants (946, 950, 954, 85, 86, 87) made only the figure of the trapezium, gave the notations and wrote the data.

- Some pupils drew special quadrilaterals: symmetrical trapezium, parallelogram, rhombus, or kite.

8. False beliefs, trial and error method

Problem 5 had a special character. The pupils could make some hypotheses and with their help, without statements and arguments, based on drawings and measuring on a figure, they easily could compute the correct result.

Cases

- Contestant 953 supposed that $|AB|=10$, $|CD|=5$. In this case the ratio of the sections of the diagonals was 2:1. From these quantities he computed the values of $|AM|$, $|CM|$, $|DM|$ and $|BM|$.
- I found in 12 works a false belief. The contestants supposed that they are searching the result in the domain of the positive integers. They made the prime factorization of 50 and decomposed it into two factors. From the pairs, they chose the convenient *(10; 5)* pair, and supposed that $|AB|=10$, $|CD|=5$. From these follows the correct result 250.
- Contestant 924 assumed from the figure, without arguments, that $|BM|=4$, $|MD|=9$, $|MA|=3$.
- There were other contestants too, who assumed that the diagonals of the trapezium divide each other in ratio 2:1. It was true, but they gave no arguments. This way they got the right result (919, 941, 951).
- Contestant 908 wrote: $|AB|^2+|BC|^2+|CD|^2+|DA|^2=450-4(|AM|\cdot|MC| + |MD|\cdot|MB|)=450-4\cdot|AB|\cdot|CD| = 450-200 = 250$. The result and the statements are good, but there were no arguments.
- There is an interesting trial in the work of contestant *913*. He got the equation: $\sqrt{a^2+b^2}\cdot\sqrt{(12-a)^2+(9-b)^2}=50$. Then he wrote that the solutions of this equation are $a=4$, $b=3$, because $\sqrt{16+9}\cdot\sqrt{64+36}=50$.

- Contestant 902 assumed that $a^2 = 8^2 + 6^2 \Rightarrow a = 10$, $b^2 = 6^2 + 4^2 \Rightarrow b = \sqrt{52}$, $c^2 = 4^2 + 3^2 \Rightarrow c = 5$, $d^2 = 8^2 + 3^2 \Rightarrow d = \sqrt{73}$, so he got that $a^2 + b^2 + c^2 + d^2 = 250$.

9. False interpretations

The cause of false interpretation is that the pupils do not remember punctually the definitions and the theorems. They suppose such facts which are not true, use such data which are not given.

Cases

- Contestant 917 used the height theorem of right angled triangles for the triangles *DCB* and *ABD*, but these triangles were not right angled.
- Contestants 916, 929 and 926 badly remembered certain theorems.
- Contestants 936 and 916 supposed the principle of symmetry where there was no symmetry ($|MA| \cdot |MC| = |MB| \cdot |MD|$).
- Contestant 944 thought that the median of the trapezium was going through the intersection point of the diagonals.

Conclusions

In geometry it is very useful to know and apply a lot of theorems; sometimes the contest problems are special cases of some, less known theorems. The decisive step in the solving of geometrical problems may be to make a figure and to introduce appropriate auxiliary lines. Generally it is necessary to give the figure with notations in formulating of a problem. The analytic geometry is helpful to describe a problem pictorially. A figure usually makes it easier to assimilate the relevant data and to notice relationships and dependences. The introduction of a coordinate system helps to solve a geometrical problem by the way of algebra, helps to compute the values. The tunnel method is an attempt to solve the problem when the student is an ace of solution of a problem. The presence of algebraic symmetry in a problem usually provides a mean for reducing the amount of work in arriving at a solution. It is obvious that without conception contestants could not complete the solution of a problem. If they try to do some routine procedure it is not sufficient for the success. Students made a false interpretation when they were absent-minded, when their knowledge was not sure. They used false relations if something was wrong in their thinking (recognizing, remembering, regrouping, supplementing). It was a very frequent false belief of the contestants that they have to solve algebraic problems in the domain of positive integers.

I was surprised that all of the right solutions of Problem 5 were different. The more experienced problem solvers had their own way to approach to the solution. If we apply the principle of learning problem solving through solving problems with the help of the best pupils' solutions for training the contestants, it would be more effective and more vivid than a tutor's interpretation. We found that the cooperative learning completes efficiently the competitive learning of the contestants.

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Problems in mathematical problem-solving

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Abstract: *Traditional mathematics education in Hungary follows the Pólya-model. Hungarian textbooks teach students using this method and professors of mathematics education analyse its applications. So it seemed to be evident for me to use it to examine mistakes in problem-solving. Besides the concepts of Pólya, I use the ideas of a Hungarian psychologist in our paper. I try to connect these concepts in order to describe mistakes in mathematical problem-solving. I hope that combining the two theories I can give a guidance to Hungarian mathematics teachers that they can easily use in their everyday work. At the end of the article there is a short description of another application of this idea.*

1. Definitions

The notions of ***problem*** and ***task*** in psychology differ from each other. The latter notion means a more general concept. The situation is similar in the theory of mathematics education. “A task is said to be a problem if its solution requires a person to combine data previously known in a way that is new (to him). If he can immediately recognize the measures that are needed to complete the task, it is a routine task (or a standard task or an exercise) for him.” Psychology words this very similarly: “The characteristic of a problem is that we have to reach a goal, namely the solution, which is unknown when the question is posed.” The word *problem* has another meaning, too.

A psychologist can use it to denote a critical situation or a mistake. The only instance when I use this meaning of the word is in the first word of the title.

By the word *thinking* I mean ***problem-solving thinking***, which is the act of deducing new connections from the known data of the problem. The reason for this usage is that I use only this kind of thinking in my examination. What I call ***problem-solving*** is not the activity, but the thinking process that leads to the final result of the problem. In my point of view ***mistakes*** are wrong results which come from wrong thinking steps.

2. Problems in Hungarian mathematics education

There are great traditions of dealing with gifted children in Hungary. Many mathematics competitions are organized and two mathematics periodicals (ABACUS, KÖMAL) are issued for students from the age of 8 (year 3) to the age of 18 (year 12). For this reason the best students of Hungary in mathematics get good results at the International Mathematics Olympiads.

But the average level of Hungarian mathematics education is not as reassuring. The Hungarian survey MONITOR showed that the difference between schools has been increasing for decades. The international survey PISA confirmed the fact that there are huge differences between the best and worst schools. (The difference among Hungarian students in their reading comprehension skills and mathematical skills is 71 %. The same ratio of students of the OECD-countries is only 36 %.) As the good students in mathematics are among the best in the world, a Hungarian teacher can increase the efficiency of education by dealing with less talented students more successfully.

It is clear that unsuccessful students are characterized by wrong problem-solving procedures. Therefore my aim is to devise a system that describes mistakes by categorising. If teachers were aware of the kind of mistakes their students' made, they could easily eliminate them. My goal is the examination of mistakes in problem-solving. For this reason I will survey the history of Hungarian theories concerning the mathematical mistakes of students.

3. Research of mathematical mistakes in Hungary

Teachers must have been aware of the importance of mistakes for hundreds of years. They could discover that certain mistakes of certain students could repeat year by year. But scholars described this phenomenon only at the end of the 19th century.

Beke Manó wrote about his experience in 1900. According to him every mathematical mistake can be traced back to false or thoughtless analogy.

Szenes Adolf examined mistakes in using the four rules of arithmetic. He claimed that these were the results of students' attention being low, or focusing on something else.

Szeliánszky Ferenc highlighted some factors as the origin of mistakes. Such are misunderstanding, lack of knowledge, copying, cheating and so on.

Faragó László found in an experiment that the cause of mistakes is the violation of fundamental principles in education.

According to *Mosonyi Kálmán* the reason of mistakes can be wrong analogy, formalism, custom, unclear notions, deficient preliminary knowledge and terminology of mathematics.

Majoros Mária observed the use of symbols in the work of her students. She used the mathematical language of her students to trace mistakes in their thinking. Theories sketched above do not categorize mistakes on the basis of problem-solving procedures. Hence I need a different theory.

4. A psychological theory of problem-solving

I will survey the theory of Pólya first, since I would like to compare it to a suitable psychological theory. (The reasons to start from Pólya's theory are (i) every Hungarian teacher of mathematics know it; (ii) it is the only theory of problem-solving that appears in textbooks.) According to Pólya, the process of problem-solving in mathematics consists of the following four steps.

P1. Understanding the problem;

P2. Devising a plan;

P3. Carrying out the plan;

P4. Looking back.

Unfortunately, this division is not detailed sufficiently, so I cannot use it to study every small phase of the process of problem-solving. I use a related psychological idea to complete this theory. According to a Hungarian psychologist, Lénárd Ferenc, thinking consists of processes taking place on two different levels, the macrostructure and the microstructure. The phases of thinking mean steps relating to the whole thinking process. The construction of the ***macrostructure*** includes the following ***phases of thinking***:

L1. fact-finding; **L2.** modification of the problem; **L3.** suggesting a way to solve the problem **L4.** criticism; **L5.** making irrelevant observations; **L6.** wonder, delight; **L7.** annoyance; **L8.** scepticism; **L9.** giving up.

I found, when I finished a teaching experiment, in which the participant teachers were using computers for mathematics problem-solving, that in the process of problem-solving there are only 3 steps. First, I will briefly discuss some of Lénárd's and Pólya's phases.

L2.: According to Pólya, one of the most important ways of devising a plan is looking for a related, more general, more special or an analogous problem, that is, *modification of the problem*. Hence I consider this phase as part of devising a plan and suggesting a way to solve the problem.

L5.: *Making irrelevant observations* shows mistakes in suggesting a way to solve the problem. But this step was not typical in the examined school-environment.

L6.-L8.: Considering *the affective categories* would be interesting for a psychologist. I do not want to measure these, since these are not interesting for me. (Some psychologists – and I – do not consider these factors as thinking steps, either.)

L9.: When students do not finish solving the problem, then the case is *giving up*. This can be traced back to a mistake in problem-solving, so in our opinion I cannot consider this step a separate thinking step.

P3.: The operation *carrying out the plan* of Pólya does not appear in the system of Lénárd. This is because it does not qualify as thinking, but is a rather manipulative process. (Schoenfeld does not include carrying out the plan in the steps of problem-solving, either.)

I could find out a fact in connection with the remaining statements, that because of the similarities I can consider the following stages to be the same: **L1.** and **P1.**, **L3.** and **P2.**, **L4.** and **P4.**. So I denote them by **1.**, **2.** and **3.** Since Lénárd's system has a richer structure, I use Lénárd's terms. So the number **1.** means **L1.**, **2.** means **L3.** and **3.** means **L4.**

Lénárd supposed that the steps of thinking are influenced not only by the whole thinking process, but by the small surroundings of the individual steps, the parts of the so-called ***microstructure***, as well. Hence Lénárd describes the following ***operations in thinking***:

A) analysis; **B)** synthesis; **C)** abstraction; **D)** comparison; **E)** comparing abstract data; **F)** understanding relations; **G)** completion; **H)** generalization; **I)** concretization; **J)** ordering; **K)** analogy.

According to my experience six operations in thinking are enough. I will make some simplifications in order to have a disjoint system.

C): *Abstraction* emphasizes a property of a whole, which cannot be considered as an independent unit. I consider *abstraction* a special part of *analysis*.

D)-E): *Comparison* and *comparing abstract data* can be considered part of understanding relations. So I can omit these operations.

H)-I): As Lénárd considered *generalization* and *concretization* as cases of *completion*, I omit the last two operations.

I consider the six remaining operations of thinking, – *analysis*, *synthesis*, *understanding relations*, *completion*, *ordering* and *analogy* and mark them with lower case letters in alphabetical order. Thus **a)** stands for **A)**, **b)** for **B)**, **c)** for **F)**, **d)** for **G)**, **e)** for **J)** and **f)** for **K).**

Thus I can organize experiments in connection with the process of problem-solving. This fact shows that my method can describe the process of problem-solving. The system can be represented by the following table.

	a) analysis	b) synthesis	c) understanding relations	d) completion	e) ordering	f) analogy
1. fact-finding						
2. suggesting a way to solve the problem						
3. criticism						

5. Brief evaluation of the experiment

I started an experiment concerning mistakes in 1995. In this experiment I examined mistakes in mathematical problem-solving. I reviewed 1274 incorrect solutions of algebraic problems, and found 1509 mistakes. (My reason to examine algebraic mistakes was that they can be recognized easily. I believe that this fact does not influence my result.) I succeeded in placing all the mistakes in my system. This shows that my idea is applicable under school-conditions. I examined solutions written for school exercises in the first place and I very rarely completed my procedure by oral questioning – only if it was necessary.

A method suitable for categorizing mistakes should satisfy the following requirements: (1) each mistake should fit into a category in an unambiguous way; (2) categorizing should be a fairly automatic process due to easily recognisable features of the categories. I will present some typical mistakes from the exercises. (I will show only one example for each type.)

I. Examination of the macrostructure

1. Probably it is an example of incorrect fact-finding when somebody believes that lg 2 is a rational number. A student can find both rational and irrational values in the log-tables. Many students do not know which is rational among these. They find 0,301 for lg 2, and they might think that 0,301 = $\frac{301}{1000}$ and therefore lg 2 is a rational number. (The reason for this mistake may be that in Hungarian schools we only prove the irrationality of $\sqrt{2}$.)

2. The majority of mistakes originate from suggesting an incorrect way to solve the problem. Many students are surprised the fact that the next multiplication

$(ab) \cdot c = ac \cdot bc$ is not correct. (I find this type of mistake if the teachers of the participant students do not emphasize the learning of different rules.)

3. It is worth making the students accustomed to always check their work. But checking can be wrong, too. When a student compared values of $\sqrt{2}$ and $\sqrt[3]{3}$ then he worked correctly and got a good result $2^{\frac{1}{2}} < 3^{\frac{1}{3}}$. In the checking he claimed that the result is good because any power of 3 is greater than any power of 2.

II. Examination of the microstructure

a) Sometimes students decompose data incorrectly during *analysis*. For instance, they interpret $2x$ not as $x + x$, but as 2 and x. Hence they get that $2x - x = 2$.

b) When they do incorrect *synthesis*, students find false or useless connections. For example, the product $(x - y)(x^3 + x^2y + xy^2 + y^3)$ is not an appropriate form of $x^4 - y^4$ when they want to simplify the fraction $\frac{x+y}{x^4 - y^4}$.

c) An error in *understanding relations* occurs when students connect two notions incorrectly. For example, many students think that x is always greater than $-x$. They explain it by the fact that a positive number is greater than a negative one.

d) *Completion* means getting the final result using the given data and the full knowledge of relations. I show some wrong examples in the topic of identities:

$$x^m x^n = x^{mn},\ (x^m)^n = x^{m+n},\ (xy)^n = x^n + y^n \text{ and } (x^m)^n = x^{m^n}.$$

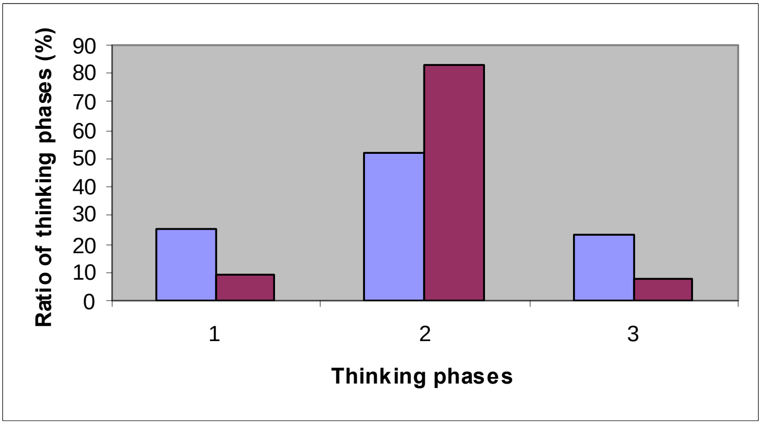
e) *Ordering* is an operation which chooses the suitable things from the group of data, notions, connections. Everyone can see that at the sorting of polynomial

$$2x^3 + 3x^2 + 4y = 5x^5 + 4y$$

a student made a serious mistake.

f) *Analogy* can be used for solving a problem which is similar to one already solved. Once students had to solve an equation in which the expression $\lg^2 5 - \lg^2 3$ occurred. Students tried to transform it similarly to the well-known expression $\lg x - \lg y$. So their result was $\lg^2 \frac{5}{3}$.

In his experiment Lénárd analysed 3426 thinking steps of 135 reports. (He observed university-students' reply after he had showed them a heavy and mysterious but not mathematical problem.) I mentioned that my system differs from Lénárd's. But in spite of the differences it is worth comparing the two findigs. (The left columns show Lénárd's data.)

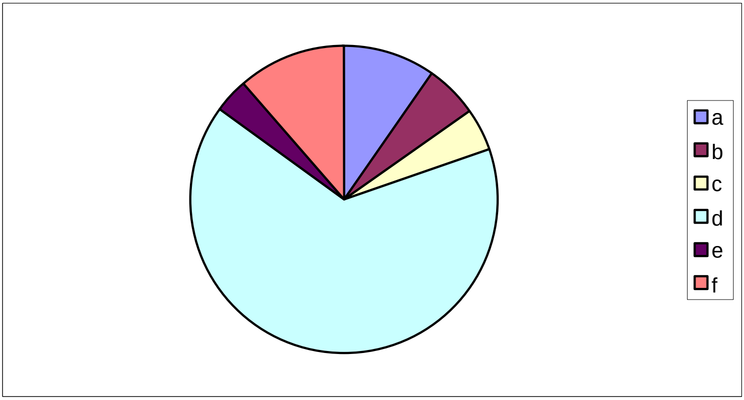


The above diagram presents many interesting facts. I will present two of them.

(i) The low ratio of mistakes in fact-finding can be a consequence of the fact that the problems themselves did not require a considerable amount of the operation.

(ii) Every mathematics teachers of Hungary can recognize the fact that the Hungarian mathematics curricula do not consider criticism as an important thing. For example this step is not compulsory after the realizing algebraic transformations.

The following chart shows results about the microstructure.



Unfortunately Lénárd did not write down his values concerning his operations in thinking, so I could not compare results his and my observations. I think that it could be explained with the Lénárd's not disjunct model of the microstructure. Everybody could see from this chart that the completion is the most important thinking operation. It refers to a bad teaching method in Hungary.

After this I will consider the two levels of thinking concurrently. The following table shows the detailed results after categorising each mistake accordingly.

	a) analysis	b) synthesis	c) understanding relations	d) completion	e) putting things and relations in order	f) analogy
1. fact-finding	31	27	24	12	31	11
2. suggestion for problem-solving	92	32	20	933	17	160
3. criticism	23	26	22	39	9	–

Analysing of this table is interesting mainly from the point of view of school-practise. But now the only fact what is important for me that my model is suitable and applicable in Hungarian education.

6. Discussion

(1) I think that it would be important to categorize the typical mistakes of all school-topics. It would be very useful for teachers who could eliminate mistakes more easily with the help of the list. (Unfortunately the detailed analysis of the surveys MONITOR and PISA are inaccessible for teachers and experts of educational institutes.)

(2) I have mentioned that the Pólya-system is too general for application in schools. However, it can be very well used in the theory of mathematical problem-solving because of the facilitative questions that illuminate the structure of problem-solving. Still, Schoenfeld and others showed that this system cannot be considered final. For instance, computer-aided mathematics teaching can raise pedagogical questions that Pólya could not have thought of. With the help of knowledge about problem-solving every Hungarian teachers can easily find problems for their students that correspond to the phases of

problem-solving. I can improve the applicability of the original theory with these supplements.

The decreasing success of mathematics teaching requires every expert to search for effective teaching methods. Perhaps the analysis of mistakes will not be the most important thing that can turn back the unfavourable processes. But I hope that this paper helps to draw attention to this neglected area that has been in great need for development for years.

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Levels of teachers' listening in working with open problems

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Abstract: *Teachers' questions and pupils' responses form an essential part of a lesson in school. Especially in the use of open-ended problems, communication between the teacher and his pupils is in a key position. Mathematics is not about getting answers, but about developing pupils' insight into relationships and structures. While the role of communication in classroom cannot be overemphasized it has to be noticed that the level of teachers' listening matters. Here we will develop a hierarchic structure to classify teachers' listening.*

Introduction

The purpose of school education in each country is, more or less, to develop independent, self-initiative, critical thinking, motivated and many-sided skilled individuals who will manage in societal settings which they will encounter later on in their life. Therefore, the key question is what kind of school instruction is optimal for this goal.

Conventional school teaching has been accused that it considers the action and the context where learning happens totally different and neutral concerning the topic to be learned. However, psychological studies show that learning is strongly situation-connected (e.g. Brown & al. 1989, Collins & al. 1989, Bereiter 1990). Furthermore, recent psychological research (Bereiter & Scardamalia 1996) has confirmed the hypotheses set e.g. by Anderson (1980) that learning of facts and procedures happens through different mechanisms.

This points out that in instruction there should be offered to pupils different methods to learn, on one hand conceptual knowledge (as facts), and on the other hand procedural knowledge (as using facts). Conventional school teaching suits very well for learning of facts, whereas learning of procedural knowledge demands pupils' self-initiated active studying. One possible solution for the latter case is offered by open learning environments, since within them one can deal with real, existing problems, be active and learn in natural settings. Since learning happens then by investigating and looking for solutions of problems, such an active studying is explained to lead to better understanding of key principles and concepts. Active learning puts pupils into a realistic and contextual problem solving environment, and thus can combine the phenomena of the real life and the class room (Blumenfeld & al. 1991).

AN OPPORTUNITY FOR CHANGE: USE OF OPEN-ENDED PROBLEMS

When looking for a new teaching method that might confront the challenges set by constructivism, the so-called open approach has been developed in the 1970's in Japan (e.g. Becker & Shimada 1997, Nohda 2000). Internationally it is accepted that open-ended problems form a useful tool when developing mathematics teaching in school in such a way that emphasizes understanding and creativity (e.g. Nohda 1991, Silver 1993, Stacey 1995, Rehlich & Zimmermann 2004). Papers from a larger group of international specialists are collected and published in a report (Pehkonen 1997).

What are Open-ended Problems?

Tasks are said to be open, if their starting or goal situation is not exactly given (cf. Pehkonen 1995). Pupils are given freedom in solving the task, which in practice means that they may end with different, but equally right solutions, depending on their additional selections and emphasis done during their solution processes. Therefore, open tasks have usually several right answers. When using open tasks in mathematics teaching, pupils have an opportunity to act like a creative mathematician (cf. Brown 1997). Open-ended problems are such open tasks that can be counted as problems. For more on open-ended problems see e.g. Pehkonen (2004).

Several types of problems are collected under the title ”open problems” (cf. Pehkonen 1995): *investigations* (a starting point is given), *problem posing* (or problem finding or problem formulating), *real-life situations* (they have their roots in the everyday life), *projects* (larger study entities, requiring independent work), *problem fields* (or problem sequences or problem domains; a collection of contextually connected problems), *problems without a question*, and *problem variations* (”what-if”-method). Several examples of different types of open problems can be found e.g. in the published papers of Nohda (1991), Stacey (1995), Silver (1995), Schupp (2002), Rehlich & Zimmermann (2004) and in the edited collection of Pehkonen (1997).

Problem fields may be described as structured investigations. One key characteristic of problem fields is that they are not bound to a fixed class grade, but are suitable for mathematics teaching from primary level to teacher in-service education. The role of the easier problems in problem fields is especially to reinforce the problem solving persistence of pupils. The most important aspect of all in these problems is the way in which they are introduced to a class: The problem field ought to be given gradually to pupils, and the continuation should be related to the pupils' solutions. Instead of the answers and results, the process of problem solving is of paramount importance. The most important aspect is the use of pupils' own creative power. Thus, the direction and scope to which the teacher expands a problem field, depends on the pupils' answers. See more on the use of problem fields e.g. in Pehkonen (2001).

Use of open problems

When using open problems in the class, the teacher should let pupils enough time and mulling ground. One way might be to split a problem into pieces, and to deal only with one piece at the end of one lesson. And the rest can be left to pupils as a home work. Thus pupils have enough time to think on the problem and to discuss together, if needed, on its solutions.

In order to use open problems in the class, there is a demand for teachers to change their teaching style. If we use the language Schroeder & Lester (1989), we might say that today most of teachers teach something *about* problem solving. But the teaching philosophy of open problems means that they should teach *via* problem solving.

THEORETICAL BACKGROUND

Teachers' questions and pupils' responses are essential parts of a lesson in school. Traditionally, it is thought that a pupil's answer shows explicitly what he/she knows. This has led to the situation that pupils expect the teacher to look for a correct answer that they expect to be in their teacher's mind. The constructivist idea, however, emphasizes that it is the teacher's task to help pupils in constructing their knowledge and understanding of concepts and mathematical thinking (cf. Davis & al. 1990).

Social-cultural research has emphasized studies on classroom discourse (e.g. Hufferd-Ackles & al. 2004). The main goal of talking mathematics in classroom is to understand and extend one's own thinking as well as the thinking of others. When the teacher wants to pay attention to his/her pupils' understanding and thinking process he/she has to listen carefully and interpretatively to the pupils.

Our aim is to concentrate on communication in mathematics teaching, and especially on teachers' listening skills. One basic demand for genuine exchange of ideas is that the participants are listening with understanding to each other. Therefore, we decided to study how mathematics teachers actually listen to their pupils.

Teachers' listening

When using discussion as a teaching method, teachers are not always listening carefully what their pupils are saying, but having their own presentation in the first place in their mind (cf. Pehkonen & Ejersbo 2004). This phenomenon is also known from earlier research in the form of teachers neglecting to use such answers that do not fit into their instructional plans (e.g. Perkkilä 2003).

Listening has been in the center of communication research more than fifty years (cf. Stewart 1983, Burley-Allen 1995), but in mathematics education it has a shorter history. Today one may find some studies on communication in mathematics with the focus on listening, among them: Davis (1997) reported a collaborative research project with a middle school mathematics teacher, and gave some examples how the teacher listens. He described the teacher's evaluative listening, interpretative listening and hermeneutic listening in mathematic lessons in eight-grade class. Furthermore, Coles (2001) has analyzed one teacher's mathematics lessons, using Davis' (1997) levels of listening. His observation was that both the teacher’s and his pupils’ listening developed with such teaching strategies that slowed down situations in the class, and offered room for discussions.

Peressini & Knuth (1998) made a thorough analysis of a teacher’s discourse in his high-school mathematics classroom and of an educator’s discourse in a university mathematics education classroom. The researchers were impressed how the teacher strived to listen to his pupils and to make sense of what they were saying and the thinking that grounded their mathematical discourse. Nicol (1999) explored prospective teachers’ learning to teach mathematics. She analyzed during mathematics lessons how elementary student teachers asked questions, how and what they listened and how they responded to pupils’ answers. As a consequence, she points out teachers’ difficulties, challenges and tensions in listening while teaching.

EMPIRICAL STUDY

We are starting a new research project the aim of which is to find out on which level teachers in the Finnish comprehensive school (grades 1–9) listen to their pupils’ answers during mathematics lessons. In order to find interesting research questions and to develop analysis methods for our further results, we carried out this pilot study. Especially we try to develop a proper taxonomy for the levels of listening.

In the pilot study we have looked through some videotaped mathematics lessons from different Finnish teachers at grade five and eight, in order to find out how they listen to their pupils during normal mathematics classes. On the basis of the literature, i.a. Stewart (1983), Burley-Allen (1995), Davis (1997), and applying our own experiences as teachers and teacher educators we formed the following classification structure that contained five levels of listening from a pupil's point of view: 1) Not listening, 2) Listening selectively, 3) Evaluative listening, 4) Interpretative listening, 5) Empathic listening (cf. Pehkonen & Ahtee 2004).

For an observer, it is not easy to interpret the levels of teachers' listening, since thinking happens in teachers' mind. Selective listening means that the teacher listens a part of a pupil's answer, but not all. Evaluative listening contains the teacher's evaluation on the correctness of a pupil's answer, i.e. its compatibility to the teacher's "correct" answer. In interpretative listening the teacher strives to understand a pupil's answer in his/her own framework, i.e. as a mathematics teacher, and to interpret it in a positive spirit.

Empathic listening differs from interpretative listening in that now the teacher tries to understand and value a pupil's ideas, although they might be strange and new to the teacher. Then the pupil and the teacher try to understand the topic from a new view point. Empathic listening has been critized impossible to implement (e.g. Stewart 1983), since then the listener should be able to switch-off his/her own feelings and thinking. Therefore, we have decided to call the highest level in the classification structure as open listening.

RESULTS

We looked the videotapes first alone, and then the lessons were transcribed. After that we picked up together some typical episodes on two-way communication. First we classified the episodes separately, and after that discussed together long enough so that we ended with the classification shown here. In the following we present with the aid of some episodes on which level teachers are listening to their pupils. However, during the classification we noticed that the classification structure is too rough, and therefore, we use sub-classes in some cases.

Description of listening levels

In the following, the levels of listening with their sub-classes are described more carefully, and they will be used later to analyze the communication episodes given. The hierarchy of listening levels is based on teachers' level of awareness and thinking. We have tried to describe the levels of listening so exactly that also other persons could apply the classification structure and reach similar results.

1. Not listening

A teacher's non-listening is surely typical in almost all lessons. The teacher often ignores what he/she hears, because he/she wants to proceed with the topic, and because a pupil's question or comment may lead to a side-track for a long time. Or he/she may have in his/her mind an idea he/she wants immediately to present to pupils, and therefore, he/she will not ponder what the meaning of the pupil's question or comment would be. Especially during lessons in middle school, there are often situations when pupils are making improper comments, in order to get others' attention. As a consequence on this level of listening, we concluded to extract two sub-levels: Firstly, the teacher does not even hear pupils' comments or questions; he/she *hears without listening* (1a). Secondly, the teacher hears pupils' comment or question, but he/she *ignores* (1b) the utterance.

2. Selective listening

The teacher is trimmed to listen only the questions concerning the topic to be dealt with. For example, an inexperienced teacher tries to listen only to such pupils from whom he/she may expect "correct" answers. He/she experiences often all kind of disturbance as a threat for his/her teacherhood. Such a teacher's behaviour has connections with strong control, discipline requirements and defence mechanisms.

3. Evaluative listening

Often a teacher has in his/her mind an answer (the model answer) which he/she expects from the pupils and with which he/she compares the pupils' answers. Therefore, the teacher's evaluation can be a simple verbal accepting utterance, as Right or Good, or it could be also only a short nod, a head's shake or a break before moving to the next question. In a more elaborated evaluative listening, the teacher comments the pupil's answer, e.g. by transforming the terms and expressions used into a correct form. Thus, we will separate a *simple evaluative listening* (3a) and a *more elaborated evaluative listening* (3b).

4. Interpretative listening

In the interpretative listening, a teacher interprets and understands a pupil's answer within his/her own thinking. He/she does not have the model answer in his/her mind. For example, he/she may repeat the pupil's answer with other words; thus he/she processes the pupil's answer, and therefore, interprets it. Also here we may separate a *simple interpretation* (4a) and a *more elaborated interpretation* (4b).

5. Open listening

Here a teacher strives to understand a pupil's thoughts from his/her world, and not only to place them into his/her own "model thinking". Open listening requires a conscious effort from the teacher to hear, follow and understand the pupil's ideas. This level represents the most open situation from the pupil's viewpoint – the pupil is not expected to think in a certain way, but he/she has freedom to develop his/her own new ideas.

Examples of listening levels

In the following, we give some episodes that are selected from the videoed lessons and represent the two-way communication in the class: usually the teacher asks, and the pupils answer. The episodes are selected so that different levels are represented as many-sided as possible. The episodes are analyzed, and the teacher's level of listening coded and reasoned.

Episode 1

Here the pupils are independently solving problems from the textbook. The teacher checks one problem together with the whole class, in order to ensure that everybody knows what to do. After that he tells the class to continue solving problems.

1 Teacher: *Now, do the problem C.*

2 Carl: *Where is it?*

3 Teacher: *It is on the page 23.*

4 Peter: *How does it go?*

5 The teacher starts to go around the class looking at the pupils' working.

Here the teacher first clearly listened to Carl's question (2), but not any more to Peter's question (4). The first one might be selective listening (level 2), whereas in the case of the second pupil he/she ignored the question (4); thus listening happens on the level 1b.

Episode 2

The topic of the lesson was fractions and their transformations. Instruction goes forward with short questions and answers.

1 The teacher writes the fraction 20/8 on the blackboard.

2 Teacher: *Can we transform 20/8 to a mixed number?*

3 Heli: *Yes*.

4 Teacher: *How many wholes will it give?*

5 Lina: *2*

6 Teacher: *And how many parts are left?*

7 Simon: *2*

8 The teacher writes 2 2/8 on the blackboard.

9 Teacher: *And can we do something to this number?*

10 Sam: *4/8*

11 The teacher corrects 2 4/8 on the blackboard.

12 Teacher: *Sorry. A good remark.*

Heli's answer (3) to the first question was straight according to the teacher's expectation. Therefore, we could suppose that the teacher is acting on the level of evaluative listening, but using a simple evaluation (level 3a). Listening of the next answer (5) can be placed also on the same level (level 3a). In the case of Simon's answer (7), the teacher automatically accepts it without thinking; therefore, we conclude that she was hearing Simon's answer (7) without really listening (level 1a). Sam answers to the teacher's question (9) by correcting the mistake made by the teacher on the blackboard. Therefore, the teacher has to connect Sam's answer (10) with her earlier question (6), and thus she has listened to Sam on a higher level (level 4b).

Episode 3

Now decimal numbers are dealt with in the lesson.

1 Teacher: *Is it allowed to add zeros after the decimal point wherever?*

2 Maria: *Yes, to the end.*

3 Teacher: *Yes. Not between wherever. To the end one can add zeros appropriately.*

The teacher accepts on Maria's answer (2) and sharpens it. This example shows a more elaborated evaluation (level 3b).

Episode 4

This is a part of a mathematics lesson on grade 8, where the topic is the use of percentages. In the episode we will find many-level listening.

1 Teacher: *If 12 hens from the henhouse are on the yard eating, and 60 % are inside, how many hens altogether are there in the henhouse?*

2 Tina: *You cannot say so that 60 % are inside.*

3 Teacher: *How many percents of the hens are on the yard, if 60 % are inside?*

4 Tina: *Ask again.*

5 Teacher: *If 60 % of the hens are inside, how many percents are then outside?*

6 Jane: *They are 12.*

7 Teacher: *In percents?*

8 Jane: *40*

9 Tina: *How can you change like that?*

10 Teacher: *Well Jane. How did you get that 40?*

11 Jane: *Well, if there are 60 % inside, then there are 40 % left from the whole 100 %.*

12 Teacher: *Yes. Now we know, how many hens there are. Now we will continue.*

The teacher changed her first question (1) into a simpler form, when Tina announced in her first comment (2) that she did not understand the question at all.

Then Tina asked the teacher to repeat her question (4). Jane gives firstly the number of the hens as an answer (6). Jane's second answer (8) was quitted only with a nod. Finally the teacher asked Jane to reason, how she had concluded her answer (10). Thus the teacher got little by little the answer in the form she wanted (11).

In this episode, the teacher's listening was mainly on the levels of evaluation and interpretation: Based on the answer (2) she interprets that Tina did not understand the situation at all (level 4a). The teacher has no problem to interpret Tina's comment; the experienced teacher divided her question into two parts. In the case of the answers (6 and 8), listening is evaluative (level 3a). The teacher's reaction to Tina's comment (9) shows that the teacher interpreted her answer as non-understanding (level 4a), and therefore, asked Jane to give reasons to her answer. The teacher's last listening to Jane's answer (11) was simple evaluative (level 3a).

DISCUSSION

The classification of teachers' listening into five main levels is based on videotaped mathematics lessons; we ended up with the classification structure given in Figure 1. In practice we noticed that the five-step classification structure for teachers' listening was too coarse, and we divided some levels into two sub-levels.

Not listening hearing without listening ignoring
Selective listening

Evaluative listening
simple evaluation
more elaborated evaluation
Interpretative listening
simple interpretation
more elaborated interpretation
Open listening

Figure 1. The hierarchic classification structure of teachers' listening to their pupils.

In our pilot study that contained only ten teachers and from each 1–3 lessons, teachers' listening was mainly on the levels 1, 2 and 3a. Only in some cases, the teachers reached the level 3b, and very rarely the level of listening seemed to come up to the level 4 (interpretation). In our data, there was no case that could have been classified on the level 5 (open listening). It is worth noticing that in our episodes we show examples from each level of teachers' listening that we were able to find in the lessons, and therefore, it is no way representative.

According to our considerations the levels (1)–(3), i.e. up to evaluative listening, seem to belong to the conventional style of teaching. There the teacher delivers knowledge to be learned, and checks whether his/her pupils have adopted it. In this teaching model, it is not so important what pupils are thinking as whether they have adopted the information. Also earlier research results show that the level of a teacher's listening to his/her pupils seems to be in the best cases evaluative (e.g. Pehkonen & Ejersbo 2004).

The listening levels (4)–(5) pertain to the constructivist style of teaching. Here the point is, what kind of ideas pupils have of the topic to be learned, and whether their subjective knowledge is compatible to the objective knowledge. Learning conception compatible to constructivism would demand that teachers listen to their pupils also on the levels of interpretative and open listening. Until then pupils begin to pay attention to their own conceptions and their deviation from the way shown in mathematics. For the teacher, open listening is absolutely important, in order he/she could perceive how his/her pupils interpret matters and what kind of difficulties pupils might have in understanding the topic. Thus he/she gets hints via which he/she can help pupils to check their conceptions and thinking.

A final note

Listening belongs to teachers' pedagogical skills, but it has been paid fairly little attention in teacher education programs. For example, if a teacher is not careful enough, he/she can develop a habit to use selective listening and concentrate only on listening to correct answers. Consequently, pupils are led to schematic thinking. The communication between the teacher and his/her pupils will improve, when the teacher shows that he/she tries to understand what the pupils mean. Then the pupils are more ready to cooperate, i.e. to express also their spontaneous thoughts. The teacher's task is no more to be sure that the pupils understand subject matters in a certain way, but the aim is to make a change possible in the pupils' thinking. Emphasizing authority or embarrassing pupils will inhibit the formation of free communication situations.

The question of teachers' levels of listening needs more research, since this seems to be in a key position when implementing open teaching methods. Especially interesting it would be to investigate the levels of teachers' listening in lessons with different working methods as well as in different school forms, in order to reach a reliable description of the state-of-art in teachers’ listening. Our next step will be to collect from different grade levels a big sample of videotapes, and to apply them the classification structure of listening levels.

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Intransitive Structures

Simulation of the creation of a mathematical theory

The conception of the "Hamburger Model for fostering high talented students" outlined by an example

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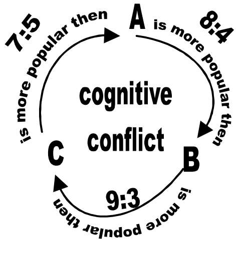
Abstract: *This paper treats of a special concept of problem-oriented learning processes which we call "simulation of the creation of a mathematical theory". This term should accentuate the main idea of our concept in order to show differences and similarities to a large range of topics which are called "problem-solving". In chapter 2 you found a short description of the concept. A paradigmatic example completes this description and helps economically to point out the sense and may give an implicit answer of more questions (adequate to our concept which too includes implicit learning processes).*

1. Introduction: Spotlights on a wide problem field

a) Elections

Twelve persons want to vote for one of three candidates. The table shows the individual order of preference of the voters, „3-2-1" (3 is the best).

	Twelve orders of preference											
A	3	3	3	3	3	1	1	1	1	2	2	2
B	2	2	2	2	2	3	3	3	3	1	1	1
C	1	1	1	1	1	2	2	2	2	3	3	3



How can one make such tables?

Which is the smallest similar structure?

Look for more extreme examples!

b) Optimal distribution of numbers on dices

Consider the following game: Every player creates a dice with the sum S=30 of all numbers on its faces. The goal is to make a good distribution of this numbers with a high probability to get a higher number when throwing the dice.

Some questions to stimulate the creation of a theory (and examples for valid theorems)

Does exist a dice which can not lose?

Indeed, there is one but not {5,5,5,5,5,5) , but {0,2,4,6,8,10}.

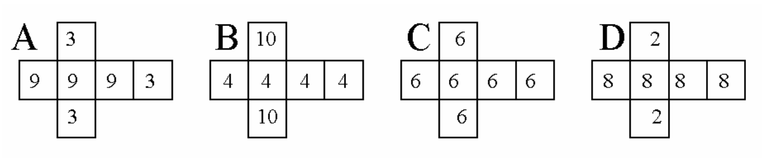
Does exist for all sums S a dice which cannot lose?

No, only for the sums 1,2,3,4,5,6,7,8,10,12,15,18,30,36.

Let us think of k dices. What‘s about „dices” with more than k faces?

If $S>k^2$ *there is no „best dice“, otherwise there is one for some special S.*

c) Efron’s set of non-transitive-dices



To figure out whether there are rich and not too complicate structures in this problem field (which can be discovered by students from age 16 to 20), I made first a data-mining-tour by using a computer.

Leading-questions to stimulate the creation of a mathematical theory:

What is the highest winning-probability $\mathbf{P_{max}}$ which you can realize at the weakest point in an Efron-circle of k dices with n faces?

What is the smallest number of dices k_{min} which you can realize P_{max}?

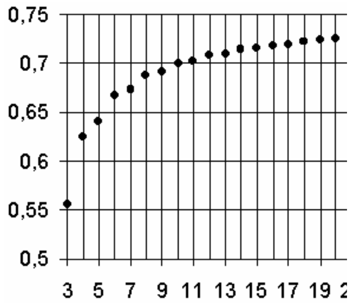
What is the highest probability $P_{max}(3)$ for winning which you can realize at the weakest point in an Efron-circle of 3 n-sided dices?

Some results of experimental mathematics

n	win : loose	P_{max}	k_{min}	$P_{max}(3)$
3	5 : 4	0,556	3	0,556
4	10 : 6	0,625	4	0,5
5	16 : 9	0,64	5	0,6
6	24 : 12	0,667	4	0,555
7	33 : 16	0,673	5	0,571
8	44 : 20	0,688	6	0,609
9	56 : 25	0,691	7	0,555
10	70 : 30	0,7	6	0,6
11	85 : 36	0,702	7	0,595
12	102 : 42	0,708	8	0,583
13	120 : 49	0,71	9	0,615
14	140 : 56	0,714	8	0,586
15	161 : 64	0,716	9	0,6
16	184 : 72	0,719	8	0,609
17	208 : 81	0,72	9	0,588
18	234 : 90	0,722	10	0,611
19	261 : 100	0,723	9	0,601
20	290 : 110	0,725	10	0,6
21	320 : 121	0,726	11	0,616
22	352 : 132	0,727	10	0,595
23	385 : 144	0,728	11	0,608
24	420 : 156	0,729	12	0,609
25	456 : 169	0,73	13	0,6
26	494 : 182	0,731	12	0,615
27	533 : 196	0,731	13	0,603
28	574 : 210	0,732	12	0,607
29	616 : 225	0,732	13	0,614

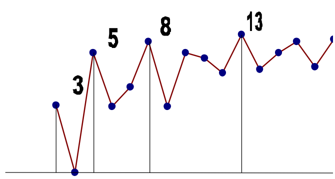
In this table are some **eye-catching patterns**:

P_{max}



The sequence seems to be convergent with limit 0.75.

$P_{max}(3)$



The fibonacci-numbers occur here as a sequence of local extrema, increasing in a monotone way.

2. Didactic framework - three qualitatively different levels and their impact on the student's activities

In the „Hamburger Projekt“ of fostering high talented students in the past 20 years a broadly based set of material has been developed for an arrangement of lessons, oriented on structures of real processes of mathematical invention. The material can be used in “ordinary school lessons”, too.

The students' activity includes phases of exploring (i.e. investigation of special cases, seeking for patterns), constructive expansion of their present mental network (by an interplay of guesswork, proofs and refutations), phases of generalisation in different dimensions (e.g. by changing prevailing conditions) and working on own subjects that arise during work (this last point is a very important characteristic).

We call these structural elements “simulation of the creation of a mathematical theory”. The “face of the created theory” is not predetermined by the teacher but it is a product of a discussion-process based on partnership.
We prefer relevant elementary problem fields with aesthetic and relevant structures (the latter in the sense of a “clear teacher-conscience”).

Our approach has many connections to POLYAS' (1957) ideas about heuristic processes. But the processes we try to initiate are embedded in a larger frame and the training of heuristics is more implicit (e. g. we do not use sequences of problems around a guiding principle of pre-selected heuristics as found in the program SINUS (BAPTIST 1998)).

The process includes elements of communication (as described by LAKATOS (1976)) and requires a minimum age (our students are at least 16 year old). Metacognition is a controversial subject that we therefore use only very carefully and only if obviously necessary.

The following table compares three different concepts of teaching mathematics and their impact on the kind of activities (I think that all three of them are essential at the right time, the problem is to find a well balanced mixture).

Learning situation	**Characterisation of Students' activities**
Optimised presentation of math. Very often in lessons at universities and school and in articles or books. Terms are predetermined, „optimised system" of definitions, theorems... Simple exercise.	To large extend only reproduction. Individual cognitive styles are not considered in the majority of cases. Many routine-tasks to assure „instrumental understanding" (ref. to SKEMP 1978). No genetic growth of concept formation, often only memorizing networks of terms.
Problem-solving i.e. in competitions like Math.-Olympics or at university (small exercise-groups), „genetic procedure" locally, but sometimes results appear as if they were created by a card trick. Sometimes not typical for mathematics; „short fibres". Authoritarian frame.	It needs more autonomy and ideas. Training of heuristics. Explicit given target, less necessity to build a network of terms, to analyse proofs in order to generalize (if this is not explicit claimed and often it is not because of difficult evaluation).
Simulation of the creation of a theory Open problem-fields. Strive for connections. Genetic learning, needs power of persistence (not possible with too young students) typical for evolution of mathematics.	Many own decisions, individual focus on special aspects and continuative questions. Own creation of terms (later tuning for communication), analysis of proofs. Intrinsic motivation is essential.

Learning situation	**Teachers main-role and -activities**
Optimised presentation of math. To be found in lessons at university and school and in articles or books. Terms are predetermined, „optimised system” of definitions, theorems... Simple exercise.	Lectures. Detailed planning of the learning process. T. „keeps the pot boiling“, often by extrinsic motivation
Problem-solving i.e. in competitions like Math.-Olympics or at university (small exercise-groups), „local genetic“, but sometimes like a card trick. Sometimes not typical for mathematics. „short fibres“. Authoritarian frame	In competitions: Choice of problems, evaluation of the solutions. In courses also: To give hints, to create a „good sequence“ of problems (sometimes oriented on heuristics‘). etc...
Simulation of the creation of a theory Open problem-fields. Strive for connections. Genetic learning, needs staying-power (not possible with too young students) typical for evolution of mathematic.	moderation, Dialog partner. Role model.

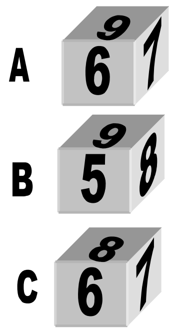
In chapter 3 we show examples of the various activities of our students. It is an interesting observation that dispositions occurred to represent the Efron-circles in different ways. We found different kinds of mathematical cast of mind as defined by KRUTETSKII (1976). Combined with different levels of abstractive capabilities we get a 2-dimensional matrix in which the problem solving activities of our students can be sorted.

3. Problem solving and the creation of a theory, examples from the work in our groups

The impetus at the beginning

The dices **A = (1,2,5,6,7,9), B = (1,3,4,5,8,9), C = (2,3,4,6,7,8)**

build up a strange set. If you play at dice for a higher number you will find out, that – under statistical aspects – **B is better than C**. The second comparison under statistical aspects of A and B shows that **A is better than B**. Normally we would expect, that A is better then C. But contrary to expectations we find out, that C is better then A.



First of all:
Prove the statement (with paper and pencil or by playing at real dices with the given numbers). Identify the 3 ratios of gain and loss.

Continuation:
Construct a similar strange set of dices with a more extreme ratio of gain and loss.

Expansion:

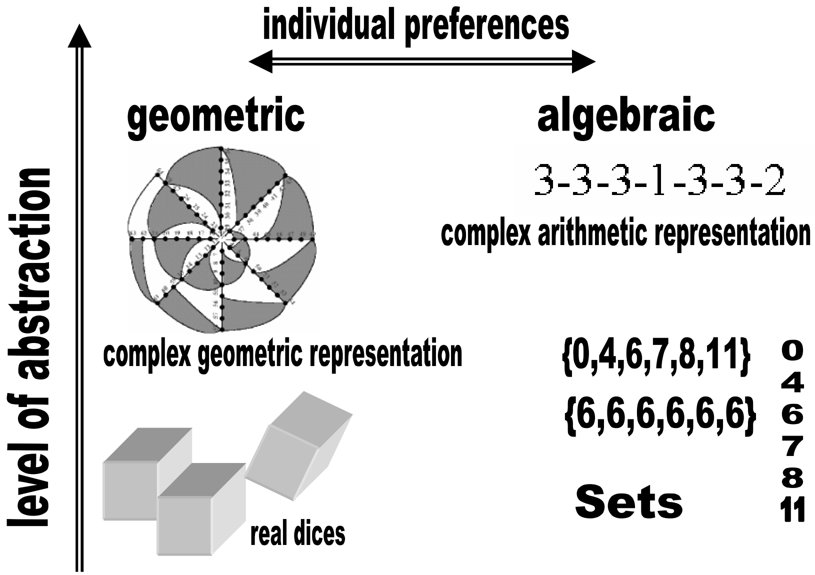
(1) What's about „dices" with a smaller or higher number of sites (i.e. tetrahedron or octahedron or n-sided wheel of Fortune)?

(2) More than 3 dices?

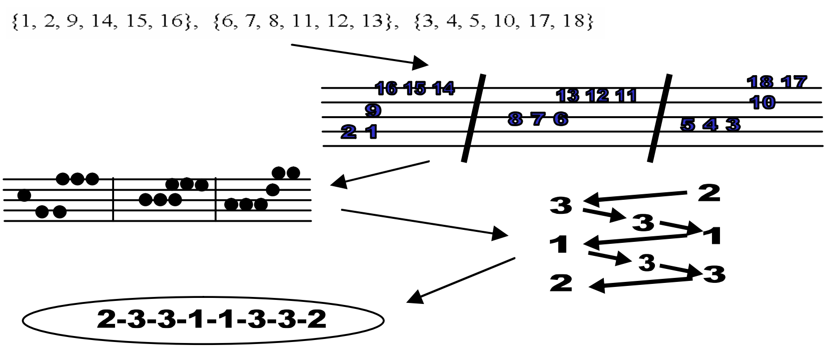
(3) ...Your own ideas

Representations used in different ways

The used representations are sorted in two dimensions:

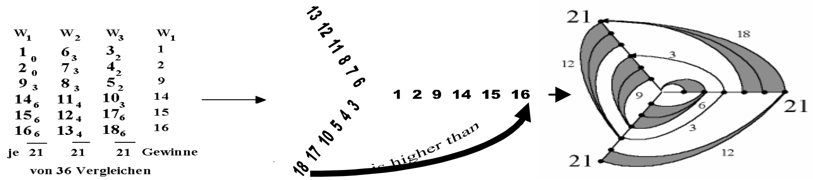


A process of metamorphosis from concret algebraic representation to an abstract algebraic representation, combined with analogy (this process took a long time):



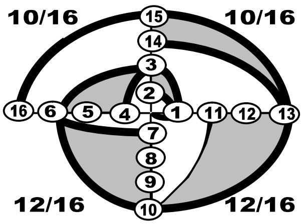
New object includes all <u>relevant</u> information

In this picture we go into the opposite direction; it shows a metamorphosis from concrete algebraic representation to an abstract geometric representation:



Some elements of the theory we created here:

Some students used the complex geometric representation. Here is an example of considerations by using this representation which helped to create the following strong theorem.



Paradigmatic example

In every set of dices ordered in a circle- is a "main-spiral". (Proof: transitivity in IN.)

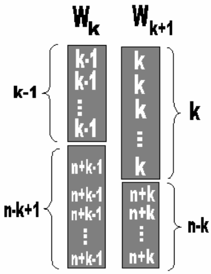
Every main-spiral represents many sets of dices. The best of all Efron-dices ("efronbest") can be constructed in the way presented to the left. (Proof: it is easy to see.)

This method of constructing the main-spiral (one tour-around, then decrease height every step for 1), produces Efron-circles with the highest probability at the weakest point. (Proof: not too simple.)

Result: We have an universal geometric method to build up the best (n,n)-Efroncircles which are possible.

Now we have a look of an analogical result using an algebraic representation:

We start with W_1=(n, n, ... n). By k steps of improvement we get the dice W_{k+1}. k faces of this dice are marked with the number k and on top of n-k faces we have the number n-k. This dice is compared with W_k. W_1 is compared with W_n.



The number G of combinations for winnings is

$G = k\cdot(k-1) + n\cdot(n-k) = k^2 - (n+1)\cdot k + n^2$

G(k) is a equation of a parabola

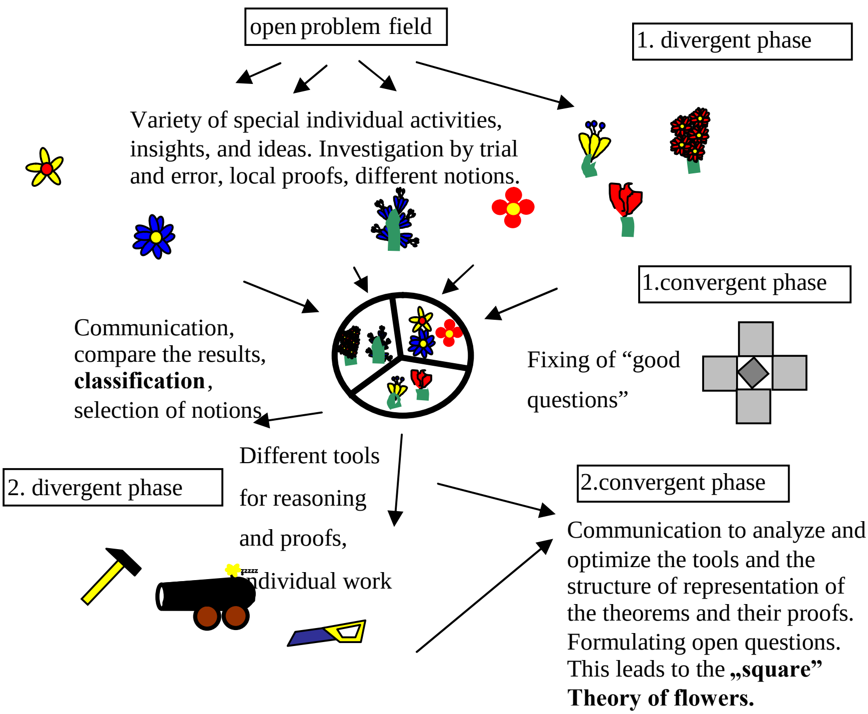
The minimum is found at $k = (n+1)/2$

So we get at the weakest point $\mathbf{G}_{\min} = \left(\frac{\mathbf{n}+1}{2}\right)^2 - (\mathbf{n}+1)\cdot\frac{\mathbf{n}+1}{2} + \mathbf{n}^2 = \mathbf{n}^2 - \frac{(\mathbf{n}+1)^2}{4}$.

By this the following theorem is proved:

The lowest probability for winning in the constructed W(n,n)-Efron-circles is convergent to p = 0,75. (For even n it can be shown in a similar way.)

4. The meta-structure of working



5. Final conclusions

Making good lessons is a very complex problem. The teacher has to take into account different needs of all persons involved in the lesson at the same time, as well as different goals which might contradict each other. Taking learning as a process of social construction, we have to be aware that it depends essentially on a mixture of all components being involved. Therefore, it is very difficult to make an appropriate external evaluation. Therefore, this article does not aim to give a fixed recipe for a "good" lesson that should basically be left in the teacher's responsibility. Instead, we want to describe concepts that have worked well in certain settings. The description of our concepts is designed to illuminate a special open approach in the large field of problem oriented teaching.

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An everyday life problem related to expected values

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Abstract: *Today there is a change in Hungarian school-mathematics. Following the trends of PISA it has to pose much more weight on the real world problems. There is a challenge to bring in nearer the old famous Hungarian mathematics tradition and the provocations of the modern world. We consider the following problem:*

"Suppose there was one of six prizes inside your favorite box of cereal. Perhaps it's a pen, a plastic movie character, or a picture card. How many boxes of cereal would you expect to have to buy, to get all six prizes?" First we build a model for this situation. After solving the problem on two different ways (one of them is known the second is suitable to answer other questions) we will generalize the problem in two directions: the number and the distribution of prizes.

0. Introduction

In this article we develop further an interesting advanced problem the so called *cereal box problem*, see [W99] or [GV02] and on the internet: www.mste.uiuc.edu/reese/cereal/cerealWilkins.pdf. The mentioned homepage gives possibilities to organize lessons dealt with this problem by different simulations and their observation. The mathematical background is not easy. We try to show it because there are two different ways which are not equal complicate. The citied source analyse only one and does not investigate with the generalization, which is an important part of mathematical thinking and problem solving. In our case there are natural situation with another assumptions.

We show another proof for the statement about expected value and thereafter generalize first on different number of prizes and later in the direction to the other distribution of the prizes (not equally distributed prizes). The idea of using thieve-formula came from A. Grétzy [GV02]. I developed it in to the direction of different numbers of prizes and using it to answer of questions about the distribution for example maximal probability (mode) and median of the distribution and finally to calculate the expected value on the other way.

I. Modelling

We are starting to build a model. If there are six prizes and all of them are equally distributed, then buying one box can be seen as a throw with one (regular) dice. Of course this model can be modified in the case of unequally distributed prizes. In this case we can use a computer random simulator with a given distributed six numbers. The question is, how many times the dice have to be thrown until the event "every six numbers came out". The minimal number is 6 (if different numbers come out by every throwing), and of course rarely very big numbers of throwing is possible. For the first solution of the problem we develop the probability distribution of the number of the needed throwing. In order to do it we use the logical thieve formula. How many different positions does exist, that the nth throw happens first that every six number came out? We have to choose one number, which firstly occurs at the *n*th throw that can be done 6 times. The rest 5 different numbers occurred during the (*n*-1) throw, but we count such cases also where only 4 different numbers came out, we have to lose these etc. Consequently we get the following formula:

$6(5^{n-1}-5\cdot 4^{n-1}+10\cdot 3^{n-1}-10\cdot 2^{n-1}+5\cdot 1^{n-1})$, where *n* is positive whole number at least 6. Using the well-known fraction for the Laplace-probability we get:

$$p_n=\frac{6(5^{n-1}-5\cdot 4^{n-1}+10\cdot 3^{n-1}-10\cdot 2^{n-1}+5\cdot 1^{n-1})}{6^n}=$$
$$=\left(\frac{5}{6}\right)^{n-1}-5\left(\frac{4}{6}\right)^{n-1}+10\left(\frac{3}{6}\right)^{n-1}-10\left(\frac{2}{6}\right)^{n-1}+5\left(\frac{1}{6}\right)^{n-1}$$

This is linear combination of five different geometrical series. We can research the monotony character of this distribution. By a graphical calculator the first members of this distribution can be determined:

p_6=0,01543210	p_{15}=0,06136739	p_{24}=0,01465061
p_7=0,03858025	p_{16}=0,05379166	p_{25}=0,01228270
p_8=0,06001372	p_{17}=0,04662805	p_{26}=0,01028488
p_9=0,07501715	p_{18}=0,04007466	p_{27}=0,00860364
p_{10}=0,08276892	p_{19}=0,03421596	p_{28}=0,00719165
p_{11}=0,08439429	p_{20}=0,02906446	p_{29}=0,00600768
p_{12}=0,08160926	p_{21}=0,02458994	p_{30}=0,00509617
p_{13}=0,07604251	p_{22}=0,02073905	p_{31}=0,00418665
p_{14}=0,06898715	p_{23}=0,01744802	p_{32}=0,00349122

Table 1

It shows that the probabilities are monotone increasing at the start and later change into monotone decreasing. This fact gives the idea comparing two neighbour members by the following way:

$$\left(\frac{5}{6}\right)^{n-1}-5\left(\frac{4}{6}\right)^{n-1}+10\left(\frac{3}{6}\right)^{n-1}-10\left(\frac{2}{6}\right)^{n-1}+5\left(\frac{1}{6}\right)^{n-1}<\left(\frac{5}{6}\right)^{n}-5\left(\frac{4}{6}\right)^{n}+10\left(\frac{3}{6}\right)^{n}-10\left(\frac{2}{6}\right)^{n}+5\left(\frac{1}{6}\right)^{n}$$

We get the inequality:

$$\frac{1}{6}\left(\frac{5}{6}\right)^{n-1}+5\left(\frac{3}{6}\right)^{n-1}+\frac{25}{6}\left(\frac{1}{6}\right)^{n-1}<\frac{5}{3}\left(\frac{4}{6}\right)^{n-1}+\frac{20}{3}\left(\frac{2}{6}\right)^{n-1}$$

Multiply by 6^n we get:

$$5^{n-1}+30\cdot 3^{n-1}+25<10\cdot 4^{n-1}+40\cdot 2^{n-1}$$

If $5^{n-1}>10\cdot 4^{n-1}$ and $30\cdot 3^{n-1}>40\cdot 2^{n-1}$ then it is sure that the inequality turns consequently it is a good estimation if we solve:

$$5^{n-2}>2\cdot 4^{n-1}=8\cdot 4^{n-2} \qquad 3^n>4\cdot 2^{n-1}=2\cdot 2^n$$

$$1{,}25^{n-2}>8 \qquad \text{and} \qquad 1{,}5^n>2$$

$$n>2+\frac{\lg 8}{\lg 1{,}25}\approx 11{,}31 \qquad n>\frac{\lg 2}{\lg 1{,}5}\approx 1{,}71$$

From it follows if n =11 or 12 then turns the inequality. We have to control only these values and we get the wished result: the maximum is the p_{11}, which can be seen in the table 1 as well.

In the next page the median will be counting after the mode (maximal probability of the distribution).

For this goal we have to add the probabilities till the sum reaches 0,5.

$$\sum_{n=6}^{m}\left[\left(\frac{5}{6}\right)^{n-1}-5\left(\frac{4}{6}\right)^{n-1}+10\left(\frac{3}{6}\right)^{n-1}-10\left(\frac{2}{6}\right)^{n-1}+5\left(\frac{1}{6}\right)^{n-1}\right]=$$

$$=\left(\frac{5}{6}\right)^{5}\left[\frac{1-\left(\frac{5}{6}\right)^{m-5}}{\frac{1}{6}}\right]-5\left(\frac{4}{6}\right)^{5}\left[\frac{1-\left(\frac{4}{6}\right)^{m-5}}{\frac{2}{6}}\right]+10\left(\frac{3}{6}\right)^{5}\left[\frac{1-\left(\frac{3}{6}\right)^{m-5}}{\frac{3}{6}}\right]-$$

$$-10\left(\frac{2}{6}\right)^{5}\left[\frac{1-\left(\frac{2}{6}\right)^{m-5}}{\frac{4}{6}}\right]+5\left(\frac{1}{6}\right)^{5}\left[\frac{1-\left(\frac{1}{6}\right)^{m-5}}{\frac{5}{6}}\right]=$$

$$=6\left(\frac{5}{6}\right)^5\left[1-\left(\frac{5}{6}\right)^{m-5}\right]-15\left(\frac{4}{6}\right)^5\left[1-\left(\frac{4}{6}\right)^{m-5}\right]+20\left(\frac{3}{6}\right)^5\left[1-\left(\frac{3}{6}\right)^{m-5}\right]-$$

$$-15\left(\frac{2}{6}\right)^5\left[1-\left(\frac{2}{6}\right)^{m-5}\right]+6\left(\frac{1}{6}\right)^5\left[1-\left(\frac{1}{6}\right)^{m-5}\right]=$$

$$=1-6\left(\frac{5}{6}\right)^m+15\left(\frac{4}{6}\right)^m-20\left(\frac{3}{6}\right)^m+15\left(\frac{2}{6}\right)^m-6\left(\frac{1}{6}\right)^m$$

The last term is the cumulative probability distribution.

At the median is this expression ca. 0,5. We estimate over the searched *m* value by the first two members of this express because the sum of the rest four members always is positive. To prove this statement it is enough to see the following inequations are right for every $m > 6$ natural number.

$$15\left(\frac{4}{6}\right)^m > 20\left(\frac{3}{6}\right)^m \text{ and } 15\left(\frac{2}{6}\right)^m > 6\left(\frac{1}{6}\right)^m$$

the first inequations is equivalent by $3\cdot 4^m > 4\cdot 3^m$ consequently by $4^{m-1} > 3^{m-1}$ which is true for every possible *m* values; similarly the second equivalent by $5\cdot 2^m > 2$, consequently by $5\cdot 2^{m-1} > 1$.

Using this estimation we get $1-6\left(\frac{5}{6}\right)^m \geq 0{,}5$ from that $\left(\frac{5}{6}\right)^m \leq \frac{1}{12}$ which fulfils if

$$\left(\frac{6}{5}\right)^m \geq 12 \text{ and so } m \geq \frac{\lg 12}{\lg 1{,}2} = \frac{\lg 10 + \lg 1{,}2}{\lg 1{,}2} = \frac{1}{\lg 1{,}2}+1 \approx 13{,}6.$$

It means that the median is sure less or equal then 14.

Controlling this result by the table 1 of probabilities which shows that the sum of the members from p_6 to p_{12} is less then 0,5 but adding p_{13} will be more then 0,5. It is not so bad estimation.

Another question is: maximum how many boxes have to be bought for example with at least 0,95 or 0,99 probability. Again used the last estimation for the sum of the probabilities: $1-6\left(\frac{5}{6}\right)^m \geq 0{,}95$ and so $\left(\frac{5}{6}\right)^m \leq \frac{0{,}05}{6}$ or $\left(\frac{5}{6}\right)^{m-1} \leq 0{,}01$.

It means that $\left(\frac{6}{5}\right)^{m-1} \geq 100$ consequently $m \geq 1+\frac{\lg 100}{\lg 1{,}2} = 1+\frac{2}{\lg 1{,}2} \approx 26{,}258$.

The estimated value is 27, which shows again a good result controlled by the exact value getting from the table 1 27, exactly the same. Similarly with 0,99 we get the value *m* > 36,07 it means *m*=37. The exact value is the same. If we buy 37 boxes then the chance of getting the six different prizes is more than 0,99.

These questions could not be answered without the probability distribution opposite to the expected value, as we will see. Using this probability distribution the expected value is to calculate following way:

$$E(X) = \sum_{n=6}^{\infty} n\left[\left(\frac{5}{6}\right)^{n-1} - 5\left(\frac{4}{6}\right)^{n-1} + 10\left(\frac{3}{6}\right)^{n-1} - 10\left(\frac{2}{6}\right)^{n-1} + 5\left(\frac{1}{6}\right)^{n-1}\right] =$$

$$= \sum_{n=6}^{\infty} n\left(\frac{5}{6}\right)^{n-1} - 5\sum_{n=6}^{\infty} n\left(\frac{4}{6}\right)^{n-1} + 10\sum_{n=6}^{\infty} n\left(\frac{3}{6}\right)^{n-1} - 10\sum_{n=6}^{\infty} n\left(\frac{2}{6}\right)^{n-1} + 5\sum_{n=6}^{\infty} n\left(\frac{1}{6}\right)^{n-1}$$

We will calculate after each other the five members:

The first:

$$\sum_{n=6}^{\infty} n\left(\frac{5}{6}\right)^{n-1} = 6\left(\frac{5}{6}\right)^{5} + 7\left(\frac{5}{6}\right)^{6} + 8\left(\frac{5}{6}\right)^{7} + \ldots =$$

$$= 6\left[\left(\frac{5}{6}\right)^{6} + \left(\frac{5}{6}\right)^{7} + \left(\frac{5}{6}\right)^{8} + \ldots\right] + \left[\left(\frac{5}{6}\right)^{7} + \left(\frac{5}{6}\right)^{8} + \left(\frac{5}{6}\right)^{9} + \ldots\right] + \ldots =$$

$$6\left(\frac{5}{6}\right)^{5}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^{2} + \ldots\right] + \left(\frac{5}{6}\right)^{6}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^{2} + \ldots\right] + \left(\frac{5}{6}\right)^{7}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^{2} + \ldots\right] + \ldots =$$

$$= 6\left(\frac{5}{6}\right)^{5} \cdot 6 + \left(\frac{5}{6}\right)^{6} \cdot 6 + \left(\frac{5}{6}\right)^{7} \cdot 6 + \ldots =$$

$$= 36\left(\frac{5}{6}\right)^{5} + 6\left[\left(\frac{5}{6}\right)^{6} + \left(\frac{5}{6}\right)^{7} + \left(\frac{5}{6}\right)^{8} + \ldots\right] =$$

$$= 36\left(\frac{5}{6}\right)^{5} + 6\left(\frac{5}{6}\right)^{6} \cdot 6 = 36\left(\frac{5}{6}\right)^{5} + 36 \cdot \frac{5}{6}\left(\frac{5}{6}\right)^{5} = 66 \cdot \left(\frac{5}{6}\right)^{5}$$

Similarly using iteratively the formula for the geometrical series we get the following equations:

$$\sum_{n=6}^{\infty} n\left(\frac{4}{6}\right)^{n-1} = 6\left(\frac{4}{6}\right)^{5} \cdot 3 + 3\left(\frac{4}{6}\right)^{6} \cdot 3 = 24\left(\frac{4}{6}\right)^{5}$$

$$\sum_{n=6}^{\infty} n\left(\frac{3}{6}\right)^{n-1} = 6\left(\frac{3}{6}\right)^{5} \cdot 2 + 2\left(\frac{3}{6}\right)^{6} \cdot 2 = 14\left(\frac{3}{6}\right)^{5}$$

$$\sum_{n=6}^{\infty} n\left(\frac{2}{6}\right)^{n-1} = 6\left(\frac{2}{6}\right)^{5} \cdot 1{,}5 + 1{,}5\left(\frac{2}{6}\right)^{6} \cdot 1{,}5 = 9{,}75\left(\frac{2}{6}\right)^{5}$$

$$\sum_{n=6}^{\infty} n\left(\frac{1}{6}\right)^{n-1} = 6\left(\frac{1}{6}\right)^{5} \cdot 1{,}2 + 1{,}2\left(\frac{1}{6}\right)^{6} \cdot 1{,}2 = 7{,}44\left(\frac{1}{6}\right)^{5}$$

That means for the expected value:

$$E(X) = 66\left(\frac{5}{6}\right)^{5} - 120\left(\frac{4}{6}\right)^{5} + 140\left(\frac{3}{6}\right)^{5} - 97{,}5\left(\frac{2}{6}\right)^{5} + 37{,}2\left(\frac{1}{6}\right)^{5} = 14{,}7$$

Another formula is for this expected value following the way of article [W99] which use the remark that for the first prize we have to wait 1 throw (6/6), for the second prize (the probability of a new number 5/6) is 6/5, and so on for the sixth prize (the probability is 1/6) 6/1, consequently:

$$E(X) = \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 1 + 1{,}2 + 1{,}5 + 2 + 3 + 6 = 14{,}7$$

See more detailed in [W99]. This way of thinking can be visualised very good by Markov-chain and further a recursive system of equation.

II. Generalization for different number of prizes

We can generalize easy by the second method, which shows for N different prizes:

$$E(X) = \frac{N}{N} + \frac{N}{N-1} + \frac{N}{N-2} + \frac{N}{N-3} + \ldots + \frac{N}{2} + \frac{N}{1} \approx N \ln N$$

The first “distribution’ method” using the thieve-formula gives the following result, first for the number of favourable events

$$N\{(N-1)^{n-1}-(N-1)\cdot(N-2)^{n-1}+\dots\cdot 3^{n-1}+(-1)^{N-2}2^{n-1}+(-1)^{N-1}(N-1)\cdot 1^{n-1}\}.$$

Using the well-known fraction for the Laplace-probability:

$$p_n=\frac{N\{(N-1)^{n-1}-(N-1)\cdot(N-2)^{n-1}+\dots\cdot 3^{n-1}+(-1)^{N-2}2^{n-1}+(-1)^{N-1}(N-1)\cdot 1^{n-1}\}}{N^n}=$$

$$=\left(\frac{N-1}{N}\right)^{n-1}-\binom{N-1}{1}\left(\frac{N-2}{N}\right)^{n-1}+\dots+(-1)^{N-2}\binom{N-1}{N-2}\left(\frac{2}{N}\right)^{n-1}+(-1)^{N-1}N\left(\frac{1}{N}\right)^{n-1}$$

From it the mode (maximal probability) and the median of the distribution can be calculated of course with the suitable N value.

The expected value can be determined by this way following:

$$E(X)=\sum_{n=N}^{\infty}n\left[\left(\frac{N-1}{N}\right)^{n-1}-\binom{N-1}{1}\left(\frac{N-2}{N}\right)^{n-1}+\dots+(-1)^{N-2}\binom{N-1}{N-2}\left(\frac{2}{N}\right)^{n-1}+(-1)^{N-1}N\left(\frac{1}{N}\right)^{n-1}\right]$$

The sum can be counted by step for step members. This leads in the first step to the express:

$$\sum_{n=N}^{\infty}n\left(\frac{N-1}{N}\right)^{n-1}=N^2\left(\frac{N-1}{N}\right)^{N-1}+N\left(\frac{N-1}{N}\right)^{N}+N\left(\frac{N-1}{N}\right)^{N+1}+\dots$$

$$=N^2\left(\frac{N-1}{N}\right)^{N-1}+N\left\{\left(\frac{N-1}{N}\right)^{N}+\left(\frac{N-1}{N}\right)^{N+1}+\dots\right\}=$$

$$=N^2\left(\frac{N-1}{N}\right)^{N-1}+N^2\left(\frac{N-1}{N}\right)^{N}=N^2\frac{2N-1}{N}\left(\frac{N-1}{N}\right)^{N-1}=N(2N-1)\left(\frac{N-1}{N}\right)^{N-1}$$

To control the result we substitute N=6 then comes out $66\left(\frac{5}{6}\right)^5$ and it was the first member of the expected value in the case of six prizes. Second step we get

$$\sum_{n=N}^{\infty}n(N-1)\left(\frac{N-2}{N}\right)^{n-1}=N(N-1)\left(\frac{N-1}{N}\right)^{N-1}+(N+1)(N-1)\left(\frac{N-1}{N}\right)^{N}+\dots$$

Similarly to the first step this sum is equal to

$$\frac{3N^2-2N}{4}(N-1)\left(\frac{N-2}{N}\right)^{N-1}.$$

Again can be controlled by the substitution N=6.

Similarly using the formula of the endless geometric series we get the wanted result, which is too complicate to write now.

III. Generalization of not equally distributed prizes

The case of the „not equally distributed prizes" is much more complicate I can prove only for *two different prizes*:

$$1+\frac{p}{1-p}+\frac{1-p}{p}\geq 1+2=3=\frac{2}{1}+\frac{2}{2}$$ and for three different prizes:

$$1+\frac{p_1}{1-p_1}+\frac{p_2}{1-p_2}+\frac{p_3}{1-p_3}+$$
$$+\frac{p_1p_2}{p_3}\left(\frac{1}{1-p_1}+\frac{1}{1-p_2}\right)+\frac{p_1p_3}{p_2}\left(\frac{1}{1-p_1}+\frac{1}{1-p_3}\right)+\frac{p_2p_3}{p_1}\left(\frac{1}{1-p_2}+\frac{1}{1-p_3}\right)\geq 5{,}5=\frac{3}{1}+\frac{3}{2}+\frac{3}{3}$$

The first inequality (the case two prizes) is easy to be shown. For the second we have to use that $p_1+p_2+p_3=1$ $\frac{p_1}{1-p_1}+\frac{p_2}{1-p_2}+\frac{p_3}{1-p_3}\geq\frac{3}{2}$ is true, using the inequations between the harmonic and arithmetical mean:

$$\frac{p_1}{p_2+p_3}+\frac{p_2}{p_1+p_3}+\frac{p_3}{p_2+p_1}\geq\frac{3}{2}$$

$$2\geq\frac{3}{\frac{p_1}{p_2+p_3}+\frac{p_2}{p_1+p_3}+\frac{p_3}{p_2+p_1}}$$

$$\frac{\frac{p_1}{p_2+p_3}+\frac{p_2}{p_1+p_3}+\frac{p_3}{p_2+p_1}}{3}\geq\frac{3}{\frac{p_1}{p_2+p_3}+\frac{p_2}{p_1+p_3}+\frac{p_3}{p_2+p_1}}$$

but the right side

$$\frac{\frac{p_1}{p_2+p_3}+\frac{p_2}{p_1+p_3}+\frac{p_3}{p_2+p_1}}{3}\geq\frac{\frac{p_1}{p_2}+\frac{p_2}{p_1}+\frac{p_1}{p_3}+\frac{p_3}{p_1}+\frac{p_2}{p_3}+\frac{p_3}{p_2}}{3}\geq\frac{3\cdot 2}{3}=2\,.$$

It remains to prove:

$$\frac{p_1p_2}{p_3}\left(\frac{1}{1-p_1}+\frac{1}{1-p_2}\right)+\frac{p_1p_3}{p_2}\left(\frac{1}{1-p_1}+\frac{1}{1-p_3}\right)+\frac{p_2p_3}{p_1}\left(\frac{1}{1-p_2}+\frac{1}{1-p_3}\right)\geq 3$$

Starting with a new order following the same divisors:

$$\frac{1}{1-p_1}\left(\frac{p_1p_2}{p_3}+\frac{p_1p_3}{p_2}\right)+\frac{1}{1-p_2}\left(\frac{p_2p_3}{p_1}+\frac{p_1p_2}{p_3}\right)+\frac{1}{1-p_3}\left(\frac{p_1p_3}{p_2}+\frac{p_2p_3}{p_1}\right)\geq 3$$ and since

$$\frac{p_1}{1-p_1}\left(\frac{p_2}{p_3}+\frac{p_3}{p_2}\right)+\frac{p_2}{1-p_2}\left(\frac{p_3}{p_1}+\frac{p_1}{p_3}\right)+\frac{p_3}{1-p_3}\left(\frac{p_1}{p_2}+\frac{p_2}{p_1}\right)\geq 2\left(\frac{p_1}{1-p_1}+\frac{p_2}{1-p_2}+\frac{p_3}{1-p_3}\right)$$

it is enough to show, that $2\left(\frac{p_1}{1-p_1}+\frac{p_2}{1-p_2}+\frac{p_3}{1-p_3}\right)\geq 3$.

Introducing new variables for the divisors $a=1-p_1,\ b=1-p_2,\ c=1-p_3$
we get $2\left(\frac{1-a}{a}+\frac{1-b}{b}+\frac{1-c}{c}\right)\geq 3$.

It means that it is enough to see $2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)\geq 3$. From the inequality between the harmonic and arithmetic mean follows:

$\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\leq\frac{a+b+c}{3}=\frac{3-(p_1+p_2+p_3)}{3}=\frac{2}{3}$ that is equivalent with the wanted inequality.

We finished the proof with it.

The case of six different prizes is again so complicate, that not only the proof, but the size of formulate of inequalities are so rapidly increasing that can not be handled normally. This is a good mathematical research question. *I conjecture the statement is true: if the probability is not equal-distributed the expected value will be more.* It means that for the consumer would be important to allow to control of the prize-distribution. The sales have advantage if the distribution of the different prizes differs from the equal-distribution.

Remark: College Rehlich sent me a very good structured way to build up an inductive reasoning in order to show that last statement true for every *n* positive integer. Maybe we can write a new article about this generalization. The time press do not allowed me a deeper analyse this method and writing of this idea is not success during the time of corrections. It would be better to work on it common with college Rehlich.

IV. Some didactical remarks

In the year 2001 the problem has been tested in two different age-groups. For younger (13-14 years old) it was presented as a play (see in [GV02]). The pupils constructed a model and made experiences with dice. The results were collected as a statistics. They counted the arithmetic mean. It was (15,03) quite near to the expected value (14,7). It was homework twice to repeat this experiment. On the next lesson they collected the new results and the mean was 14,67. The idea of a mathematical calculation is too high for this age group, but the computer simulation is a possible, or a similar method as Engel recommended for Markov-progress can be used for them.

For the age 17-18 as well we tested this problem but in that case the mathematical methods as well with geometrical series and logical-sieve formula. They were a higher-level group in mathematics. Two or three (he used a little help from teacher) pupils could solve the problem. They had some technical problem during the counting. One of them used a recursive way and two worked a way from the probabilities using the geometrical series. We do not use any visualisation; it can be a next step using the remark of Rehlich.

The new trend in teaching mathematics is to involve the realistic problems and more concrete mathematization-process (see f. E. PISA). The above-mentioned problem is a good muster for it, a good project-idea. An advantage of this problem is the possible different accessibilities starting with experiences computer simulation and finally the different mathematical tools. The natural generalization is a further useful character of the problem. I will work on it and try to organise a new experiment in a school.

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(How many bottles have to be bought in mean in order to collect all six different coupons?) Raabe Tudástár 2002

Problem solving and problem posing revisited

Some additional steps towards theory building and outlook to possible implementation

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Abstract: *Mathematical problem solving and posing are already taken for granted as guiding ideas in mathematics education (cf. Stacey 2004). Nevertheless, there are still deficiencies and difficulties in theoretical clarification as well as in implementation in normal classroom teaching.*

We take the work of Schoenfeld 1985 and his latest suggestions Schoenfeld 2005 as a starting point for some critical remarks and questions towards a more encompassing theory of mathematical problem solving and -finding. We look for some possible relations to modern contributions to procedural and conceptual learning (Haapasalo & Kadijevich 2000), to questions of understanding (Fennema & Romberg 1999), to brain research (Spitzer 2002) and history of problem solving (Zimmermann 1991).

This framework gave us some additional possibilities for guiding and structuring implementations.

What's a theory and what's the use of it?

"Theories should help to understand a well defined domain consistently and help to pose and answer questions"

(Freudenthal 1991, 127, 128)

Do we already have a theory of problem solving and finding? Is this domain already “well defined”? In spite of considerable progress in this direction during the last twenty years – as we will try to line out this point later on – there might be still some doubts (cf. Lester 1980, p. 290).

What might be the use of a theory?

According to Boltzmann, nothing is more practical than a good theory (cf. the title of a main lecture of Sfard at the last ICME in Copenhagen, paraphrasing this well known quotation with the following title of her lecture “What can be more practical than good research?”

Let us listen to some other quite well known persons who stress the importance of theories in research:

“It is the theory which determines what is observable” (Kant 1787, Einstein 1930, Popper 1976, Bauersfeld 1977).

But where do theories come from? Of course, they do not “spring into the world like Pallas Athene from the forehead of Zeus” (cf. Feyerabend 1977, p. 362), but they develop over a long process of forming conjectures and refutations (Popper 1963), observations and their critical evaluation.

We can summarize: There is no theory without sound analysis of praxis and no structured and reflected praxis without sound theoretical considerations or, as Kant said:

“Praxis without theory is blind and theory without praxis is empty”

(Kant 1787 B XVIII)

These remarks might help to clarify, to reflect on and to understand better the theory-building process as a kind of modelling process:

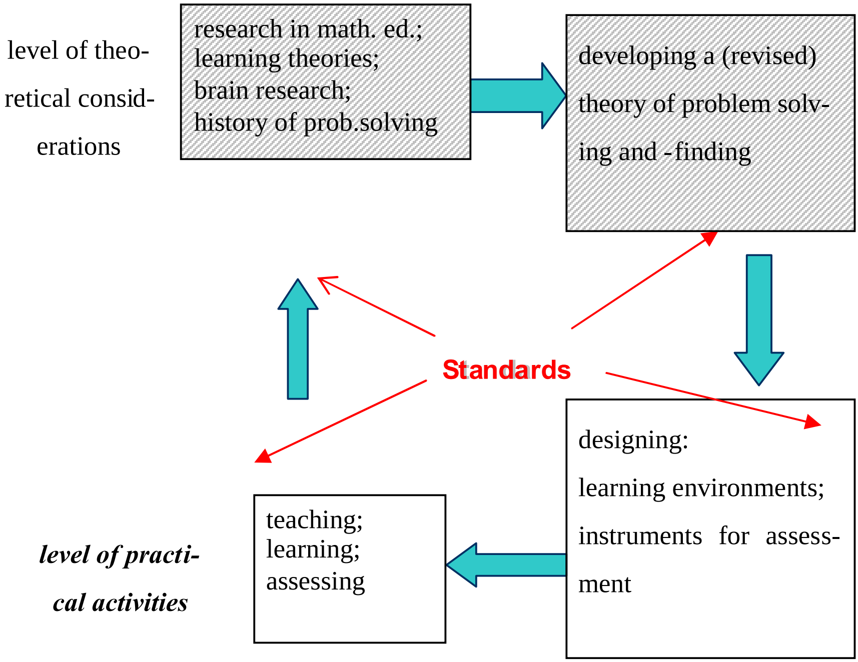

Figure 1 Process of theory formation (theory modelling)

We can start in any of the four “boxes”: on the level of praxis at the bottom or, at the level of theories on the top. In any situation, normally one wants to improve it. This means very often to follow the circle in the clockwise sense. But, sometimes it might be also reasonable, to go in the counter clockwise direction. E. g., if you want to develop a new schoolbook (box bottom right), it might be useful to have a look on latest theories (upper box to the right) etc.

Of course, all these developments and activities have to meet *standards* of quality. A good set for orientation for considerations and activities in all four “boxes” are:

Relevance, originality, validity (methods), objectivity, rigor and precision, predictability, reproducibility, relatedness (cf. Kilpatrick 1992 and Sierpinska 1992).

In the remaining paper we want to start with considerations in the "upper box to the left" and want to make a transition to the right box.

Research in mathematical problem solving and finding

Schoenfeld`s approach to problem solving 1985 and 2005

The work of Schoenfeld 1985 encompasses the following four parts:

knowledge base, strategies, beliefs, control (metacognition self-regulation).

During the last twenty years, the development of mathematics education went in directions some of which can be sketched in the following way:

- more emphasis on computer tools in the classroom (CAS, DGS, spread-sheets)
- more emphasis on *e-learning* via internet
- more interest in the learning of mathematics of *younger children*
- more focussing on *teacher training*
- more interest in *comparative studies*

These trends could be observed in discussions and presentations at ICME10.

They had and still have impact on the development of mathematical problem solving and finding, too.

In Mainz 2005 Schoenfeld presented the thesis that the following "factors" are most important to explain relevant teacher behavior:

goals, beliefs, knowledge

Taking into account these statements as well as the modern trends in mathematics education quoted above, the following questions can be posed:

a) Focus on **students**:

Schoenfeld's experience is mainly constituted by teaching college students (age about 20). There seems to be no special focus in his research on the learning and teaching of problem solving and finding of younger children. There is a special need to learn more about the role of the knowledge base and control resp. metacognition in youngsters. Too much emphasis on externalizing thought processes and reflection on them might even hamper the learning progress of youngsters (cf. Kretschmer 1984).

- At what age metacognition helps to foster problem solving and finding?
- What about individual differences in problem solving and finding styles (cf. Zimmermann 1977, 1981, Krutetskii 1976, Heinrich 2004)?
- What is the relation between modelling and problem solving and finding?
- What is the role of computers in problem solving and finding?
- What is the role of collaborative learning in problem solving and finding?
- How to assess and evaluate problem solving and –finding processes as well as understanding in a more objective and standardized way (e. g. by computer support, cf. Rehlich 2004)?
- What about cross-cultural comparative studies in problem solving and finding (cf. e.g. Cai & Hwang, 2002)?

b) Focus on **teachers**:

- What might be the possible consequences or possible answers to the questions posed in a) for teacher training?
- What about individual differences in the way of teaching problem solving and posing?

- What about relations between teachers' beliefs about problem solving and finding and their teaching methods?
- What about competencies of teachers and student teachers?
    - Flexibility/creativity with respect to mathematics AND education?
    - Sensitivity for the complexity of mathematics instruction (Kießwetter 1994, Frensch & Funke 1995, Fritzlar 2003)?
    - Design of productive learning environments (cf. Rehlich 2004)?

c) Focus on **mathematics educators**:

It is well known that there are not only different mathematical belief-systems of pupils and teachers (cf. Leder G., Pehkonen & Törner (2002), Zimmermann 1991, 1997), but also of mathematics educators (cf. Zimmermann 1983).

There is special actuality of this aspect when looking at the different mathematical background philosophies on which the problems in TIMSS (more formal) and PISA (more application oriented) were designed. When thinking, e. g. about the excellent tradition concerning mathematics in Hungary, differences in results of Hungarian pupils in these two tests (cf. Vári 2004) might be explained by the differences in the corresponding philosophies.

Relations to procedural (p) and conceptual (c) learning and understanding

According to Carpenter & Lehrer in Fennema & Romberg 1999 understanding (of mathematics) can be characterized by the following components: (1) constructing relationships, (2) extending, applying knowledge (3) reflection (4) communication.

Conceptual and procedural learning have been in focus of mathematics educators at least since Hiebert edited his nice book about this theme in 1986.

Haapasalo & Kadjievicz 2000 made an excellent update and augmentation to this field. According to them - with empirical evidence from a project about learning of fractions - simultaneous activation of p and c and of different representations (of processes or concepts) might foster better understanding (c) and problem solving processes (p). One can relate this statement to the process-competencies and content-competencies of the NCTM 2000).

Problem solving and –finding and modern brain research

There are some results from modern brain research quoted by Spitzer (2002), which support the assumption that the learning of processes is ahead of the learning of concepts. He refers to the learning of a language at early ages and especially to grammar and conceptual understanding.

By such considerations one can see a support of the following statement, too: “Cognition does not start with concepts, but rather the other way around: concepts are the result of cognitive processes.” (Freudenthal 1991, 18).

By this, one might see that the learning of heuristics should be done mainly or at least first *implicitly* (without conceptualization), cf. e.g. Bauersfeld 1993. You can find even more evidence for this thesis when looking at the history of mathematics (see next paragraph). In the book of Spitzer one can also find support of old assumptions about motivation and sustainable learning (not only of mathematics).

History of mathematical problem solving and -finding

If one looks at the history of mathematics as a very long process of development of cognitive processes, one can observe, e. g., which heuristics proved to be very "successful" over some 5000 years and which other activities and motivation helped to create new mathematics. One can find evidence that heuristics were applied nearly always implicitly over a long range of time before they were made explicit (cf. Zimmermann 1991, 2003, 2003a).

Examples for attempts to implement some elements of this framework into the mathematics classroom can be found in Scholz et al. 2005 as well as in a new text-book series (Cukrowicz & Zimmermann 2000 - 2005).

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