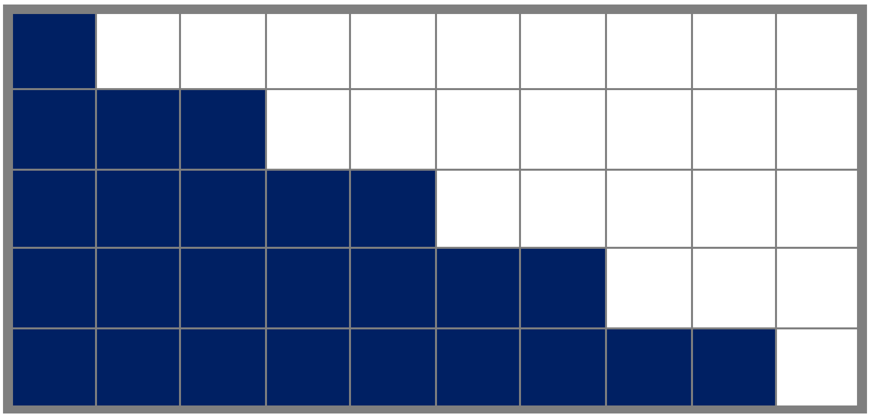
András Ambrus
Éva Vásárhelyi (Eds.)

# **Pro**blem Solving in **Math**ematics Education



Proceedings of the 19th ProMath conference
from August 30 to September 1, 2017 in Budapest

Eötvös Loránd University
Faculty of Science
Institute of Mathematics
Mathematics Teaching and Education Center

Problem Solving in Mathematics Education

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# CONTENT

# SOME PHOTOS OF THE CONFERENCE

Made by Lilla Korenova

# PROMATH 2017 – PREFACE

ProMath (**Pro**blem Solving in **Math**ematics Education) is a group of didactics experts of mathematics from all over Europe, who have the common aim of supporting and scientifically exploring problem-solving activity in mathematics among students, exploring and promoting the possibilities and preconditions of problem-solving orientation in mathematics teaching, and promoting it.

ProMath was founded by Günter Graumann (University of Bielefeld, Germany), Erkki Pehkonen (University of Helsinki, Finland) and Bernd Zimmermann (Friedrich-Schiller-University of Jena, Germany). One of the activities of this group is to organize the annual conferences since 1999.

ProMath 2017, the 19th Meeting took place at Eötvös Loránd University in Budapest, Hungary, in Pólya's home country, from August 30 to September 1, 2017. The central theme of it was chosen to be „*Problem Solving Teaching – Research and Practice*". The conference was also the annual meeting of the working group Hungary (Arbeitskreis Ungarn) of the Society of Didactics of Mathematics (Gesellschaft für Didaktik der Mathematik). A brief summary can be found on the workgroup's website: http://gdm.elte.hu/?page_id=7.

The abstracts of the individual lectures can be downloaded from the website http://promath.org/meeting2017.html.

We can look back at a diverse, inspirational and effective conference with numerous participants from Finland, Germany, Greece, Hungary, Israel, Slovakia and Turkey. I would like to thank to all participants for their valuable contributions. This volume contains the papers of the talks of the meeting. The papers of this volume are peer-reviewed according to the contents, organized by András Ambrus. For the quality of the usage of English language each author is responsible for himself/herself.

Let me mention the talks of the conference not represented by an article in this volume:

– Fatma Nur Aktas & Esra Selcen Yakici-Topbas (Ankara): Cognitive-Metacognitive Process through Mathematical Problem Solving in a Small Group: Dynamic Geometry Systems or Paper-Pencil Environments?
– Daniela Assmus, Frank Förster and Torsten Fritzlar (Halle-Wittenberg, Braunschweig): Similarities between Mathematical Problems from the Perspective of Primary Students.
– Katalin Gosztonyi (Budapest): The Role of Classroom Dialogues in the Hungarian IBME Tradition.
– Ján Gunčaga (Ružomberok): Some Aspects of Problem Solving in Historical Mathematical Textbooks.

– Tangül Kabael and Betül Yayan (Eskişehir): Preservice Middle School Mathematics Teachers' Questioning Skills in problem solving process and Their Conceptions of Problem and Problem Solving.
– Dániel Katona (Budapest): Web of Problem Threads in the Pósa Method in Hungary.
– Ana Kuzle (Potsdam): Analysis of the Development of one Teacher's knowledge for Teaching Problem Solving.
– Erkki Pehkonen (Helsinki): Developing of Teaching via Problem Posing and Solving.
– Sarina Scharnberg (Lüneburg): Qualities of Successful Problem-solving Teachers.
– Emese Vargyas (Mainz): Geometric Transformations as a Tool in Problem Solving.
– Gergely Wintsche (Budapest): The Usefulness of Independent Teacher Feedbacks in the new Mathematics Textbooks.
– Esra Selcen Yakici-Topbas & Fatma Nur Aktas (Ankara): Prospective Secondary Mathematics Teachers' Prompting in the Problem-Solving Process: The Context of Metacognitive Strategies.
– Betül Yayan (Eskişehir): Performances of Eighth Grade Students of Singapore, the United States and Turkey in Ratio and Proportion Problems.

Many thanks go to the members of the program and the organizing committee, András Ambrus, Gabriella Ambrus and Ödön Vancsó, as well as to the Mathematics Teaching and Education Center of the Eötvös Loránd University for supporting the conference and this volume. Thanks to Lilla Korenova for the photos.

Finally, I would like to express our best wishes to two prominent personalities of ProMath, Bernd Zimmermann and András Ambrus.

Out of the 19 conferences, this is the only one that Bernd Zimmermann unfortunately could not be with us because of his serious illness.

*Dear Bernd, get well soon. Everybody here is thinking of you.*

András Ambrus had his 75$^{th}$ birthday on December 19, in the middle of the period between the conference and the publication of the volume.

*Dear András, we would like to send you our congratulations and wish you good health and much happiness.*

Budapest, 2018.

Éva Vásárhelyi

# SOLVING OF REAL SITUATIONS BASED PROBLEM – EXPERIENCE WITH TEACHER TRAINING STUDENTS

Gabriella Ambrus – Eszter Kónya

Eötvös Loránd University, University of Debrecen

*An important task is for the future teachers to teach their students for the use of their mathematical knowledge even in everyday (real) situations. This task requires often from the students a change of view – according to our tests. Last year we started to work out a program in Hungary which can (may) help them to develop in this subject. Now we show some details of this program and discuss our first experience.*

## INTRODUCTION

Word problems with real context appear already in the first mathematics book in 1476 (Teviso Arithmetic) (Verschaffel et al, 2010, p. 11). In Hungary there is plenty of word problems connected to the everyday life in old arithmetic books as well, i.e. in the „Arithmetica" of Maróthi (1743). Maróthi emphasised the importance of the real world tasks[1] in the introduction of his book already. One of his goal was to give an overview about the possible applications of the topics discussed in the book, so he chose the tasks from different fields of the everyday life.

In the secondary school mathematics the Wlassics Curriculum (1899) brought a major change in Hungary which included the goal of „understanding the simple numerical relations of practical life". The curriculum emphasized the importance of computing throughout mathematics and included, among others, interest and loan calculations that are part of daily life and the use of trigonometry in surveying tasks. (Beke, 1911, Beke & Mikola, 1909, 1911). In the textbooks written by Emanuel Beke (1862-1946) (Beke was the leader of the Hungarian mathematics teaching reform around 1900), there were lot of word problems based on real life. „Students must realize how many links there are between mathematics and everyday life, sciences and our entire perception of the universe." (Beke & Mikola, 1911, p. 200).

It is important to notice that the mentioned Hungarian reality based tasks are closed i.e. there are only one possible correct solution usually.

If the task „ … requires translations between reality and mathematics what, in short, can be called mathematical modelling. By reality, we mean according to Pollak (1979), the 'rest of the world' outside mathematics including nature, society, everyday life and other scientific disciplines." (Blum & Borromeo Ferri, 2009, p. 45). Lesh and Zawojewski (2007) discussed different instructional approaches concerning problem solving. One of the existing approaches assumes that the required concepts

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[1] We use the term „task" here as an umbrella term that includes the concept of problem, too.

and procedures must be taught first and then practiced through solving routine world problems. Another approach presents students a repertoire of problem solving strategies such as „draw a diagram," „guess and check," „make a table" etc. and provides a range of non-routine problems to which these strategies can be applied. A rich alternative to these approaches is one that treats problem solving as integral to the development of an understanding of any given mathematical concept or process, mathematical modelling is one such approach. (English, & Sriraman, 2010).

Concerning the recent curriculum in Hungary[2] mathematical modelling has to be present in instruction.

So we can underline that the reality based contexts are crucial for engaging students in mathematical modelling and for preparing students to use mathematics beyond the classroom.

## THEORETICAL BACKGROUND

In the Hungarian school practice mostly closed problems (or such problems that appears at first sight to be closed) are used, (Ambrus, 2004) despite of the fact, that the open problems are highly important in the teaching of mathematics (Pehkonen, 1995; Munroe, 2015). The method „Open approach" – the use of open ended tasks on the mathematics lessons – was worked out in Japan in the 70-s of the last century. At the same time became popular the so called „tasks for researches and investigations" in England (Silver, 1995).

In a closed task the starting and goal situation is exactly given. In the open tasks the starting, the goal or eventually both situations aren't exactly given[3]. If the starting or the goal situation is not exactly given) „students may end with different but equally right solutions, depending on their additional selections and emphasis done during their solution processes." Pehkonen, 1999, p. 57) The types of open problems Pehkonen (1999) categorized as follows: (1) Investigations (the starting point is given); (2) Problem posing; (3) Real-life situations (which have their roots in the everyday life); (4) Projects; (5) Problem fields; (6) Problems without a question; (7) Problem variations („what-if-method").

Most students struggle when faced with complex and ill-structured tasks because the strategies taught in schools and universities simply require finding and applying the correct formulae or strategy to answer well-structured, algorithmic problems (Ogilvie, 2008). They are asked to solve rarely open-ended challenges, but exercises in familiar tasks, with an emphasis on completing these tasks quickly and efficiently (Schoenfeld, 1988). „The usual practice involving routine word problems, ... engages

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[2] National Core Curriculum (Nemzeti Alaptanterv, NAT) 2012.

[3]According to a more general conception, problems with exactly given starting and goal situations can be also considered as open tasks in the case when several possible way of solutions can be formulated for the task (Wiegand & Blum, 1999; Büchter & Leuders, 2005).

students in a one- or two-step process of mapping problem information onto arithmetic quantities and operations.” (English & Sriraman, 2010, p. 267). Students tend to exclude real-world knowledge and realistic considerations from their solution processes (Puchalska & Semadeni, 1987; Verschaffel, DeCorte, & Lasure, 1994; Csíkos, Kelemen, & Verschaffel, 2011).

We experienced many times in our teaching practice when the students work with real life situation where the starting situation is not exactly given that they

- can’t solve the problem;
- try to make connection between the given numbers and the question neglected the concrete problem situation;
- may not realize the openness and work with the given data;
- complete the text (automatically, „as usual”, without considering any assumption) in order to close the task and solve it obviously;
- may end it with (at least one or more) different equally correct solutions depending on additional selections, (there is an expectation to find at least one possible solution of the task connected to a possible initial assumption).

In order to better highlight our approach concerning open world problems we present here an example, the „Pocket money” problem.

## OUR PREVIOUS RESEARCH: THE „POCKET MONEY” PROBLEM

The „Pocket money” problem is a text-variation of another task “Dresses of the queen” (Ambrus & Szűcs, 2016). The variant in connection with money seemed to be more appropriate – according to the previous surveys – for students, especially for older than 4$^{th}$ grade students. (Ambrus, 2016).

*The „Pocket money” problem*
*Since Pisti[4] moved to a new house with his family, he has received his pocket money, 1000 Hungarian forints, weekly. He has saved all his pocket money since they moved in. How many days have they spent in their new home if Pisti has saved already 35 000 Hungarian forints?* (Ambrus, 2016)

At first the problem seems to be a closed whose solution is 35 times 7 which equals to 245. Nevertheless, it is an open problem as for example neither the date of the arrival of the family nor the days on which the pocket money is given to Pisti are specified in the problem. Moreover, in the problem „a week” can mean a calendar week or seven consecutive days.

Thus, for example if he receives his pocket money every Monday they must have spent 35 Mondays in their new home which means at least 34 whole weeks. Following from this, the solution can be 34x7+1=239 days. On the other hand, they could have spent at most 6 days in their new home before handing Pisti his first

[4] Hungarian boy’s first name.

(Monday) pocket money thus the family have spent at most 35x7+6=251 days in their new home. Although the problem requires not much mathematical knowledge, a solution with systematically arranged assumptions and solutions may be a challenge for 14-18 years old students as well (Ambrus, 2016).

There were several investigations with the problem „Pocket Money", between 2012 and 2016. Hungarian upper primary and secondary school students from different types of school and university students worked with the task individually without any help. The students could use as much time as they needed (usually, the solution required no more than 10-12 minutes). The main question was: Did students recognize the openness of the problem i.e. did they consider at least one assumption which were used in the solution? Our hypothesis was, that at upper grades (and at the university) most of the students can realise the openness of the task, because on the one hand the situation is easy to imagine, on the other hand the calculation needed is a routine process, so it remains time to think about the situation. The proportion of the open solutions in different groups is shown on the Figure 1 the number in the brackets shows the number of the students who took part in the investigation).

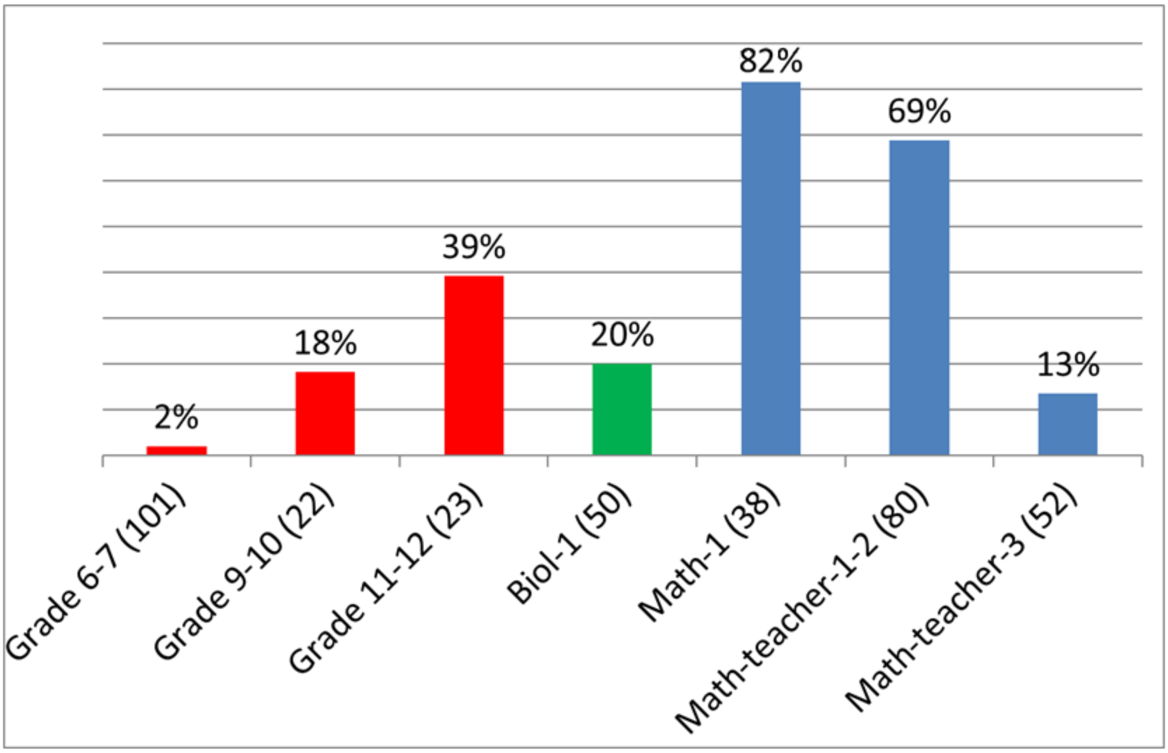


Figure 1: The open solutions of the „Pocket money" task in different groups.

The result shows a diverse picture and led us to some consideration as well. We couldn't identify clear pattern concerning the age or the mathematical interest of the students. It is obvious, that an open solution for the task not appears really often in higher classes and the interest in mathematics seems to be only one of the aspects which may have an impact on the solution. At the same time the previous school exercises and the teacher's belief could be remarkable reasons of this phenomenon. As background for the results can/must be considered (1) in what extent the students had the possibility to solve tasks of this type or (2) their belief about mathematics. Furthermore not less important factors are (3) their teacher's belief (about mathematics) and (4) the way of teaching.

Although the specific relationship between teacher's beliefs and their teaching practice is not known (Maaß, 2011, Thompson, 1992), the teachers' beliefs have a decisive influence on their students' beliefs and what is more, the image about mathematics for the students is largely decided in the school (Grigutsch, Raatz & Törner, 1998). So the belief of the teacher about mathematics (i.e. that he/she has a static or dynamic view about mathematics) influences the way of thinking (schematically or not) of their students by solving word problems.

## THE EXPERIMENTAL RESEARCH PROGRAM

With respect to the results of our previous research we concluded that the future mathematics teachers during their training have to solve open problems in real world context and have to deal with the teaching methods of such kind of problems. We elaborated a developmental program for our teacher students in order to develop their awareness towards open real world problems and to gain first experiences for a planned teacher training material concerning our Project[5].

In this paper we discuss about two research questions we formulated concerning the first teaching experiment:

1. Do teacher students recognize whether different problems are open or not?
2. How students think about the open problems?

## METHODOLOGY

We elaborated and tested the experimental program in the frame of the so called „Problem solving seminar" course, which is obligatory part of the teacher training curriculum in Hungary. The structure of the „Problem Solving Seminar" let to involve different topics in connection with problem solving practice, so using open, real world problems was absolutely in line with the aim of the course.

We tried our ideas at two universities, at the Eötvös Loránd University in Budapest and at the University of Debrecen. The number of participants in the group in Budapest was 16, while in Debrecen 18. Their main subjects are Mathematics and a

[5] Content Pedagogy Research Program of the Hungarian Academy of Sciences, 2016-2020.

second discipline, like Physics, Informatics, History, English etc. The students participated in our course were first, second or third graders.

During the semester we worked five times (out of twelve) with open reality based problems. First they solved the „Pocket money" problem; afterwards we had a 90 minutes session with (simple) open tasks in reality context. Students worked individually first with the problem, thereafter we discussed the solutions and possible questions. The further sessions took place in the first part of the seminar lessons, where we worked on the similar way with the tasks. On both universities the students worked practically with the same tasks, the seminar lecturer were the authors of this paper.

**Schedule of the course**

- Week 1-2: „Pocket money" problem – test then 4-5 simple open problems (in real world context) – discussion
- Week 3-6: 2-3 open problems besides closed tasks
- Week 7: „Party" problem – test
- Week 8-11: 2-3 open problems besides closed tasks
- Week 12: „Season ticket" problem – test

Besides the „Pocket money" problem the student had the opportunity to work alone (and write down their own solution) twice a semester. They had to solve an open reality task as well on the two usual seminar tests (among other problems related to the basic material of the subject).

Photos, notes and audio recordings were made during the sessions, in addition to the two written works.

**The test problems and their coding**

*The „Party" problem*
*Karcsi*[6] *has 5 friends and Gyuri*[7] *has 6 friends. Karcsi and Gyuri decide to give a party together. They invite all their friends. All friends are present. How many friends are there at the party?* (Verschaffel et al, 1994)

The task seems to be simple, the expected answer is: 5+6+2=13. Understanding the real situation deeper we realise that we do not know whether Karcsi and Gyuri have common friends or they are friends at all. Depending on the initial assumptions we can give more correct solutions.

*The „Season ticket" problem*
*The monthly public transport pass for students is available with any starting day and valid for 30 days including the starting day. Szilvi*[8] *buy such a pass every time. Her*

[6] Hungarian boy's first name.
[7] Hungarian boy's first name.
[8] Hungarian girl's first name.

*first pass was bought in 05.01.2015. She remembers that her pass bought in June was valid until the day she travelled to her grandmother. In which day did travel Szilvi to her grandmother in this summer?* (Vancsó & Ambrus, 2007)

At first sight the result is July 2, but some circumstances aren't specified: (1) She bought pass exactly in every 30 days? (2) She skipped some days? (3) It could happen that she had two valid passes on one day?

For the evaluation of student's written answers we used the following coding system (Table 1). This way our results can be compared with other surveys using similar type of tasks and coding.

| **First code** | **Description** | **Examples from the solutions of the „Pocket money" problem** |
|---|---|---|
| 1 | Expected answer which results from the application of the arithmetic operations elicited by the problem statement. | 7x35=245 |
| 2 | Expected answer with a technical mistake. | There is a mistake in the counting. |
| 3 | Realistic answer which follows from the effective use of real-world knowledge about the context of the task. | The real situation was at least partly considered in the solution, eg. 245-251 days or with some mistake: e.g. 245-252. |
| 4 | other answer | Eg. error resulting from misunderstanding: 35 days. |
| 0 | no answer | |
| **Second code** | **Description** | **Examples from the solutions of the „Pocket money" problem** |
| 1 | Any comments which refers to hesitation concerning the answer. | „245 days, but it can change depending on which day he receives the money"<br>„245, but I'm not sure that this is a correct answer". |
| 0 | No such a comment. | |

Table 1: Coding system for the student's solutions, on the basis of the work of Verschaffel et al. (1994).

The responses coding by 30, 31, 11, 21, 41 refers to the cases wherein a student gave a *realistic* answer to the problem Verschaffel et al. (1994).

We show some examples in the Figure 2, 3 and 4 in order to better highlight the way of coding.

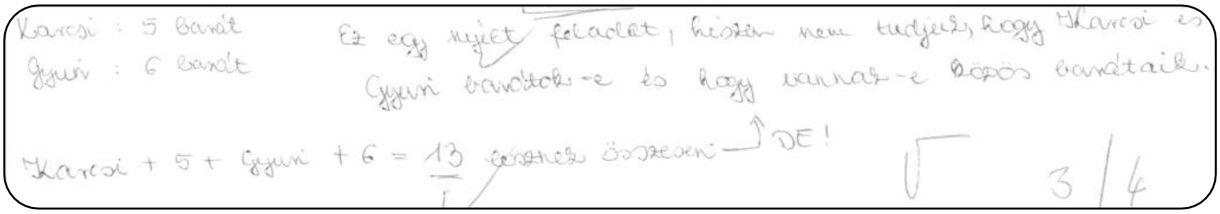


Figure 2: Solution of the „Party” problem coding by 11.[9]

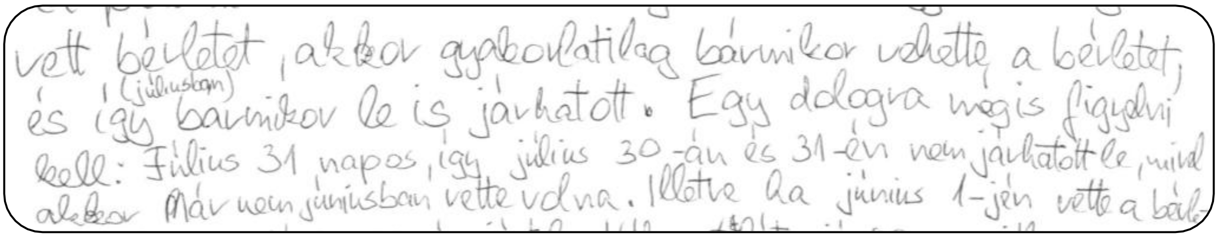


Figure 3: Solution of the “Season ticket” problem coding by 31.[10]

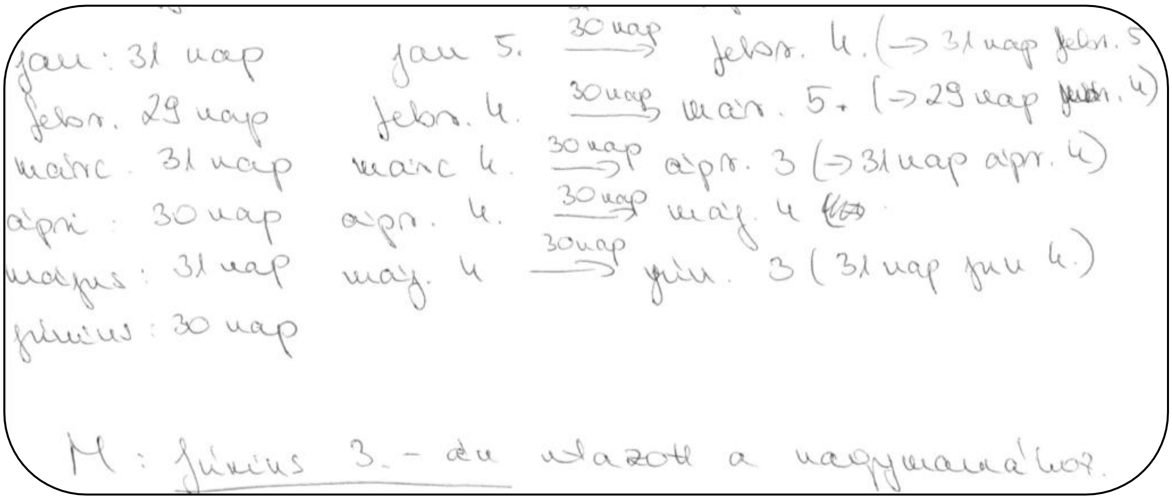


Figure 4: Solution of the „Season ticket” problem coding by 20.[11]

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[9] „The total number is Karcsi+5+Gyuri+6=13, BUT! this is an open task, because we don’t know whether Karcsi and Gyuri are friends or they have common friend at all.”

[10] „She could buy the ticket anytime, so in July it was valid in July anytime, except of 30 and 31 of July, because in this case she didn’t buy it in June.”

[11] The student calculated the wrong (answer, because he/she didn’t considered that the last pass was bought in June) date supposing – but not mentioning it – that Szilvi bought pass exactly in every 30 days.

## RESULTS

We investigated the solutions of the three open problems concerning the first research question.

Figure 5 shows the proportion of the open answers (each student produced one answer) in both groups.

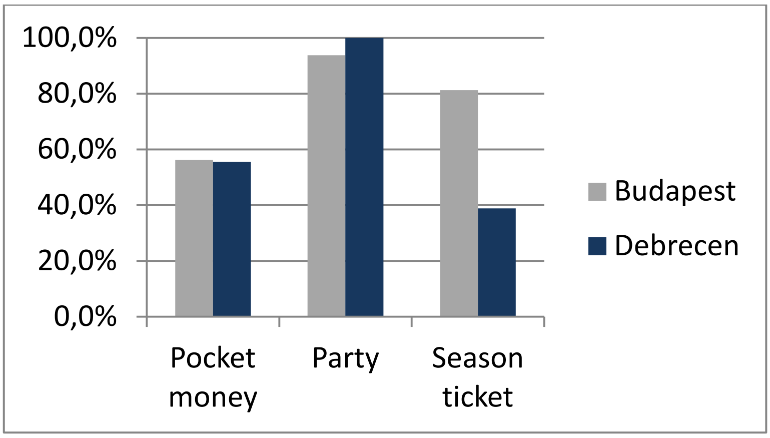



Figure 5: The open answers coding by X1 or 3X

Near to the 50% of the students recognised the openness of the „Pocket money" problem in both groups. The simple „Party" problem seemed to be open for near to the 100% of the students. The more complex „Season ticket" problem was detected as open in more cases in Budapest (BP-group) as in Debrecen (DE-group) Probably the BP-group was more familiar with this real-world situation as it turned out from the commentar of a teacher student in DE group:

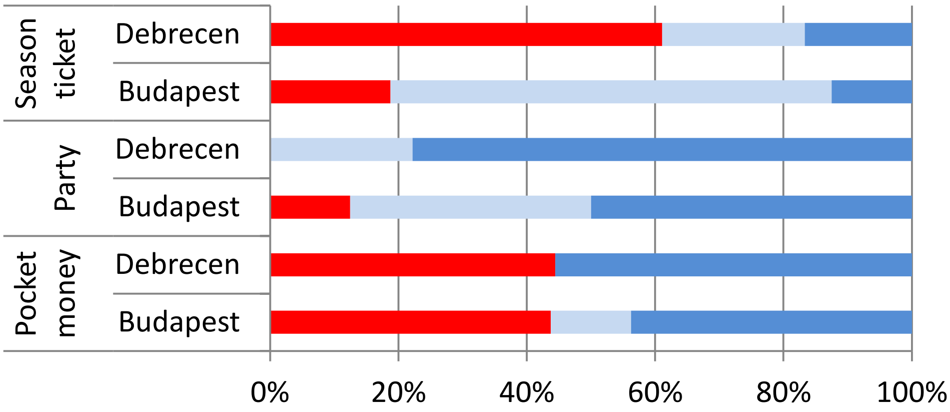
„In my opinion this is an open task. I usually buy such a ticket for the train, …, it may happen that I haven't ticket even for two weeks. …"

We classified the answers into three groups concerning their code in the following way:

- Closed answer means that the codes are 10, 20, 40.
- Correct open answer or at least partly correct open answer means that the codes are 30 or 31.
- The answer reflects for hesitation concerning the closed answer means that the codes are 11, 21, 41.

We detected similarities and differences between the achievements of the two groups (Figure 7):

- The pattern of the results of the „Pocket money" problem is similar.
- For the „Party" problem in the BP-group there exists closed answer, while in the DE-group not.
- For the „Season ticket" problem about 60% of the member of the DE-group gave closed answer. In the BP-group there are more students who formulated only hesitation and didn't give (partly) correct answer than in the DE-group.




closed: ; hesitation: ; correct (or partly correct) open

Figure 7: From the closed towards the open solution.

We also investigated the effect of our teaching concerning the recognising of the openness of the problems (realistic answers). Table 3 give some overview about the number of students whose answers fit into one of the following five categories: (1) Open-Open-Open (OOO); (2) Open-Open-Closed (OOC); (3) Open-Closed-Open (OCO); (4) Closed-Open-Open (COO); (5) Closed-Open-Closed (COC).

| | | | | | | |
|---|---|---|---|---|---|---|
| Problem | Pocket money | Open | Open | Open | Closed | Closed |
| | Party | Open | Open | Closed | Open | Open |
| | Season ticket | Open | Closed | Open | Open | Closed |
| Group | Budapest | 7 | 1 | 1 | 5 | 1 |
| | Debrecen | 4 | 6 | 0 | 3 | 5 |

Table 2: The "effect" of our teaching.

We were very happy of course that no one was in the category CCC, so everybody gave at least once open answer. We can detect development regarding the recognition of the openness of a problem in the work of those students whose answers were in the categories „COO" and „COC". The fact that 6 students from Debrecen were in the category „OOC" probably means that they were not familiar with the "Season ticket" situation in their everyday life or even the closed version of the task was too complicated for them and they got lost in the details.

At the end of the semester we discussed with the students what they are thinking about open problems and their role in the process of learning mathematics. Here we quote some of the students' opinions:

- „The tasks are interesting and useful."

- [Tasks like these] „develop thinking, so they are useful."
- [Tasks like these] „help us to formulate tasks without misunderstanding."
- [Tasks like these] „help us not to evaluate a situation rashly 'too easy'."
- „Solving tasks like these, where there are several possibilities for a correct solution, students probably don't worry if they couldn't find the only one."
- „The tasks are interesting but I don't know what they are proper for."
- „I hate this kind of tasks because there is no mathematics in them!"

As we expected, the opinions cover a wide range. Some remarks are in close connection with the concrete tasks, while others refer to the belief of the teacher students about mathematics (for example the last one). Furthermore the answers show that the viewpoint of the students is different; some of them evaluated the problems from a student's perspective, while others from the perspective of a future teacher.

## CONCLUSION

The experience with the tasks showed our students that similarly to younger pupils they think schematically time to time. It also happened that they got lost in the details or overemphasized the real situations. The tasks then the discussions helped students to recognize whether a problem is open or not. If they are familiar with a situation in their everyday life, they recognize it easier in the math problem. We agree with Cheng (2013), that „Students solutions of problems embedded in real life context often reflected their personal values and beliefs." (Chan, 2005 quoted by Cheng, 2013)

The simple problems (with simple calculations) may indicate the idea of openness, because otherwise it is too easy for them. From this experience arises the question we have to investigate in the future: What about the open problems with more mathematics?

Sometimes the teacher students hesitated even if they gave open answer, eg. „If I think that this is an open problem [the „Season ticket"], shall I also write the close solution just in case?" They are not quite sure that this kind of solution is acceptable, they are afraid that the teacher thinks on the closed version. Here we point out that solving problems is based how one thinks about mathematics. The phenomenon refers to that belief system which often appears concerning in-service teachers too. We determine as an important task to change this rigid belief system concerning the open reality based problems and to teach our future teachers how to use their mathematical knowledge even in everyday (real) situations. We emphasize that teacher students have to be familiar with the real situation based problems during the training already in order to integrate it into their prospective teacher's belief system.

As a final remark we have to mention that five sessions in a semester are enough to develop awareness of future teachers towards open real-life problems, but not enough to change their belief and behaviour concerning the math problem solving routine. This is the reason we plan further developmental program with different problems and structure in order to better understand students' thinking and the way of the effective contribution to give our students wider perspective in this topic.

**Acknowledgement**

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# THE COMPETENCES OF STUDENTS MAJORING IN TEACHER TRAINING IN MATHEMATICS PROBLEM SOLVING, LESSONS LEARNED FROM MATHEMATICSMONITOR 5 IN SLOVAKIA

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*The focus of my work is on the presentation of results of tests conducted among students of Hans Selye University majoring teacher training. The test included the tasks of official mathematics Monitor 5 of Slovak Republic for 5$^{th}$ grade students, the aim of which is to test the knowledge of mathematics acquired in the primary (1$^{st}$ to 4$^{th}$ grade). The test has been conducted among students from different years; the major aim of the test has been the evaluation of the competences of mathematics at primary level of future teachers and kindergarten teachers. Different tasks and their solutions are going to be presented in my work, special attention is paid to the solution of problematic tasks – primarily from combinatorics and compound word problems and to plausible shortcomings of cogitation of students in the curriculum of teacher training at Selye University in Slovakia and in the elementary school curriculum in Slovakia.*

## 1. INTRODUCTION

The first survey, the so-called Monitor 5 in Slovakia among students starting their 5th year focusing on their abilities and skills in mathematics was conducted in 2012. It has been done every year since then by the National Institute on Certified Surveys in Education (Národnýústavcertifikovanýchmeranívzdelávania (NÚCEM).

The major goals of the survey are:

- Survey on the knowledge and skills of students, acquisition of objective information about the achievements of students, the level of their proficiency on teaching level ISCED 2 (National Curriculum for elementary level schools classes 5-9).
- Acquisition of feedback for schools about the preparedness of students moving from ISCED1(National Curriculum for elementary level schools classes 1-4) level to ISCED 2, which can help in fostering the quality of education.
- Setting basic values necessary for the specification of added value in elementary education.

The teaching of mathematics in elementary education according to the national curriculum is based on the following factors:

The goal of mathematics teaching in the elementary level is the exploration of mathematics by games, and getting information by experiencing in line with the principle of graduality. Further essential factors are: concept making based on experiences, active knowledge of contents and concepts in mathematics, understanding texts and the specific language of mathematics, proper use of terminology.

One of the most important tasks of mathematics teaching at elementary level is to explore the relation between mathematics and reality, adopt the proper learning habits, develop skills in acquiring information individually and how to process with that information, develop skills in problem detecting and problem solving, creative thinking.

The tests have been compiled in order to be able to measure the mathematic skills of students at elementary level. The tests of Monitor 5 – 2016 were filled by students of elementary pedagogy and nursery training at SelyeJános University in the academic year 2016/2017. As we have been experiencing the lack of basic knowledge and skills of our students we have decided to conduct such a unified survey (during problem solving in mathematics classes we have witnessed large scale errors). It is unacceptable for future teachers.

**The following goals have been set for the surveys:**

- Information on the level of proficiency of students in elementary mathematics,
- Familiarity of students with concepts in elementary level mathematics,
- Success of the students in problem solving,
- Comparison of the outcome of the tests of university students with the national average of 5. Grade students,
- Confrontation of students with their own level of mathematical knowledge and their ability to adopt,
- Presentation of the current basic requirements of elementary mathematics teaching to our students,
- Confrontation of the students with the necessity of the content and experiences in education in mathematics at university level.

## 2. CONCISE DESCRIPTION OF THE TEST

The mathematics teaching according to the national curriculum *the acquisition of new knowledge based on a realistic approach, the development of the large scale of skills is done by manual and intellectual activities. The same approach is followed in the application of new mathematical knowledge in real life situations.*

The acquired mathematical knowledge sets the basis for further development as well as for their use in real life. The test has been compiled so that the academic language supervises use of the system of mathematical symbols, procedures and algorithms and helps the solution of advanced tasks. The understanding and the capability of analyzing texts including mathematical relations, tables, graphs, diagrams, and texts including numbers is essential for the solutions of tasks.

The tasks in Monitor 5 are compiled not only to measure learned knowledge but to measure cognitive skills in acquiring information, appreciative reading and logical thinking as well.
The tasks monitor the level of knowledge and skills, the capability of students to use that knowledge and to explore new problem solving strategies. The tasks are in line with the content of the national programme. The compilation of tasks and the definition of elements of knowledge is based on the advanced Bloom taxonomy:

**Cognitive knowledge**

A. *Factual knowledge* – The basic elements students must know to be acquainted with a discipline or solve problems.
B. *Conceptual knowledge* – The interrelationships among the basic elements within a larger structure that enable them to function together.
C. *Procedural knowledge* – How to do something, methods of inquiry, and criteria for using skills, algorithms, techniques, and methods.
D. *Metacognitive knowledge* – Knowledge of cognition in general, as well as awareness and knowledge of one's own cognition.

**Cognitive processes**

1. *Remember* – Recall or retrieve previous learned information
2. *Understand* – Comprehending the meaning, translation, interpolation, and interpretation of instructions and problems. State a problem in one's own words.
3. *Apply* – Use a concept in a new situation or unprompted use of an abstraction. Applies what was learned in the classroom into novel situations in the work place
4. *Analyze* – Separates material or concepts into component parts so that its organizational structure may be understood. Distinguishes between facts and inferences.
5. *Evaluate* – Make judgments about the value of ideas or materials.
6. *Create* – Builds a structure or pattern from diverse elements. Put parts together to form a whole, with emphasis on creating a new meaning or structure. (Testovanie 5, 2016)

**Testing 5 – 2016**

The test in mathematics contained 18 tasks with a content from real life and 12 tasks with mathematical context. The level of difficulty was based on the revised Bloom taxonomy from the level of remembering to the level of evaluation. The highest level of creating was not included in the tests.

During the compilation of tests authors focused on the following goals:

- the use of mother tongue and academic language,
- application of mathematical symbols,
- application of acquired definitions, procedures, algorithms for the solution of tasks,
- the use of tables, graphs and diagrams,

- the use of logic and cognitive thinking for the solution of tasks.

## 3. MONITOR 5-2016 SURVEY

According to the the Law no. 245/2008 on education the extern testing of students in elementary schools is provided by the National Institute on Certified Surveys in Education (Národný ústav certifikovaných meraní vzdelávania – NÚCEM). The testing in 2016 was conducted under the name „Testovanie 5-2016" on 23rd November 2016. The test can be downloaded here:
https://testovanie5.zones.sk/materialy/testy/2016/T5-2016_Mat_v_MJ.pdf.

### 3.1. Outcomes of the national testing

The total number of students filling the mathematics tests was 45 286, the traditional paper test was filled by 43 737 students, electronic form was chosen by 1 549 students. The total share of successful tests was 62,3%, the success rate of female students was 61,6%, of male students 62,9%. The share of public schools was 92,1%. The success rate of private schools was 69,8%, of schools run by church it was 68,4%.

Students from the schools with a small number of student where only first 4 years are taught had a lower success rate, 52,2% than the national average. (Testovanie 5, 2016)

According to the national evaluation the results have been satisfactory fulfilling former anticipations. The best results were received from tasks in the fields of numbers, variables, operations. The lowest results were seen in the fields of combinatorics, probability counting and statistics. These textual tasks contained elements of combinatorics. The results depended mostly on the student's capability of appreciative reading, correct interpretation and understanding of the text and values derived from tables, pictures and graphs.

The evaluation of the tests included various important factors: difficulty, sensitivity, statistical significance and reliability. The most important indicators of the national testing are compiled in the following table:

| | |
|---|---|
| Total number of students | 45 286 |
| Number of disabled students | 3 091 |
| Average grade of 4th year students | 1,75 |
| Average of successful students in % | 62,3 |
| Maximum points | 30 |
| Standard error in % | 24,1 |

| | |
|---|---|
| Reliability (Cronbach alfa) | 0,90 |
| Correlation of end term grade with success at tests | -0,74 |

Table 1: Outcomes and parameters of tests in mathematics (source: *http://www.nucem.sk/documents/46/testovanie_5_2016/Spr%C3%A1va_T5-2016_final.pdf*)

The attributes of the mathematical tests were perceived as normal. The difficulty of tasks ranged from 28,7% (task no. 13) to 82,7% (task no 14), the correlation coefficient ranged from 0,29 to 0,60. The easiest tasks (no. 2) were related to numbers, variables, operations and (task no.14) to geometry and measuring. The following tasks were perceived as easy (difficulty ranging from 80% to 60%): task no. 3 (77,8%) and tasks no. 07, 23, 12, 17, 22, 20, 05, 06, 30, 21, 25, 18, 19 and 28 (60,1%).

The following tasks were regarded as medium difficult (difficulty ranging from 60% to 40%), task no. 04 (59,9%) and tasks no. 26, 16, 08, 24, 01, 15, 09, 27, 11 a 10 (41,6%).

Only one task no. 3 (28,7%) was evaluated as difficult. It was related to combinatorics, probability counting and statistics.

The test reinforced that open tasks (in which the answer was a small number) and closed tasks (in which students had to choose from a multiple choice) could be well divided by several attributes and sensitivity. The sensitivity ranged from 46,0% to 85,4%.

The average rate of success in various fields:

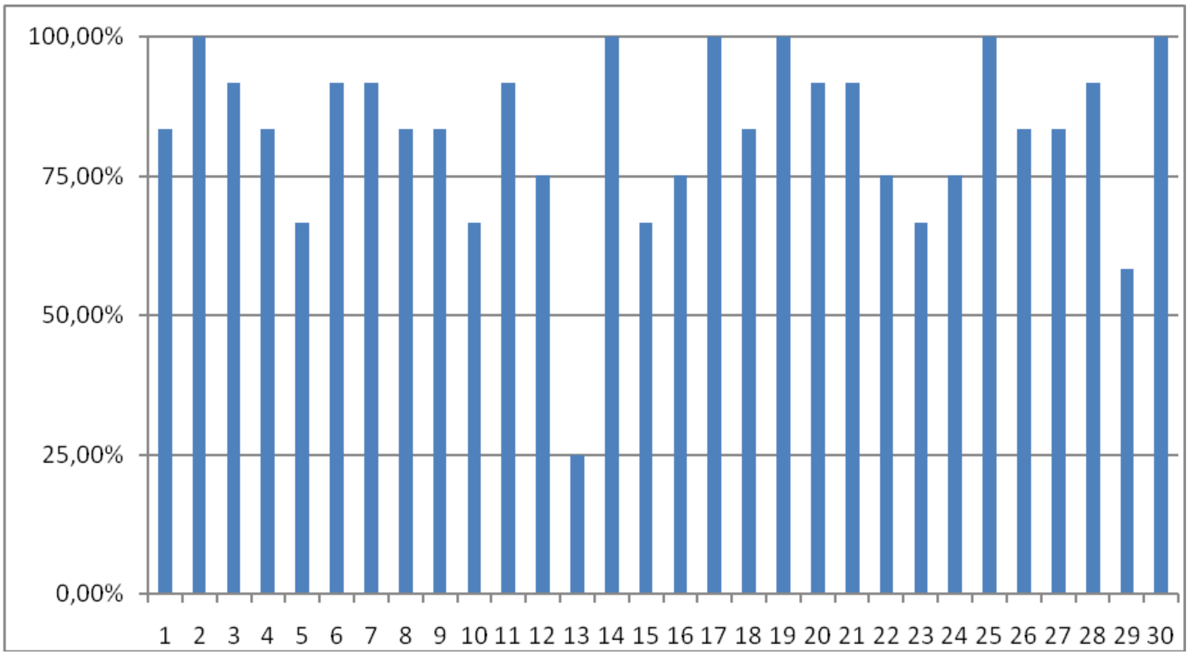
1. Numbers, variables, operations with numbers – 66,4%,
2. Sets, relations, functions, tables, diagrams – 64,2%,
3. Geometry and measuring – 58,1%,
4. Combinatorics, probability counting, statistics – 49,1%,
5. Logics, reasoning, proofs – 58,2%.

The national average of success in schools with Slovak as language of teaching was 63,2%, the national average success of schools with Hungarian as language of teaching 48,8%. The diffenerence is not significant if we take the number of students into account. The results of schools with 9 year teaching was 63,1%, schools with 4 years teaching 52,2%.

### 3.2. The results of the students of the Selye János University majoring Nursery Training and Elementary Pedagogy

I conducted a survey among the students of the Selye János University, the students did the Monitor 5 – 2016 test in the second semester of 2016/2017 term. The number of the university students in the first year was 14, out of whom 12 completed the test.

The average rate of success the test was 79,06%. Fortunately the following tasks had been solved easily by all the students: 2, 14, 17, 19, 25, 30. In the following tasks only one student out of 12 made mistakes: 3, 6, 7, 11, 20, 21 and 28, the success rate was above 90%. The task no. 13.was the most difficult, only 25% of the students were capable of solving it. The task no 29.was solved only by 58,33% of the students. Tasks no. 5, 10, 15, 23 were solved by less than 70% of the students (66,67%). The success rate of all the other tasks ranged from 70 to 90%.




Graph 1: Success rate of solving the tasks by the university students (source: own calculation).

The results of the students of Selye János University have been compared to the outcomes of the national testing conducted by NÚCEM. The aggregated outcomes according to themes are the following:

| Themes | Average of 5th year students | Students of the Selye János University – Master |
|---|---|---|
| 1. Numbers, operations with numbers | 66,4% | 83,85% |
| 2. Sets, relations, functions | 64,2% | 94,44% |
| 3. Geometry, measurements | 58,1% | 86,11% |
| 4. Combinatorics, probabilities, statistics | 49,1% | 58,33% |
| 5. Logics, reasoning, proofs | 58,2% | 79,17% |
| Aggregated outcome | 62,3% | 80,14% |

Table 2: Aggregated outcome according to themes (source: own calculation).

University students have scored better results at the tests by 15% - 30% higher than 5th year elementary school students. Alarming results were seen in combinatorics, probability counting and statistics, where the scores were almost equal. It is a distressing fact that there was only one field in which the rate of success would have been higher than 90%, the tasks were meant for primary level, not for future teachers. If we take into consideration the social background of elementary school students, their time devoted to mathematics learning, the results are worrisome as well.

The university students as well as 5th year elementary school students got their worst results in Combinatorics, probability counting and statistics. University students scored best at sets, relations, functions. Elementary school students scored best at numbers and operations.

No tasks were left out students risked guessing as well at last tasks. During the evaluation wrong answers could be deducted to miscalculation, inconsistency as well as poor knowledge of mathematical concepts.

## 4. PROBLEM SOLVING AT PRIMARY SCHOOL LEVEL

In my work I present those tasks for 1st to 4th grade students (primary school) which are considered problem solving tasks and we shall analyze the outcomes of the Monitor – 5 2016 tests. In the first part a short summary of the teaching of problem solving, problem solving skills and knowledge development in the national curricula.

The word problems are regarded as basic pillars of the teaching of problem solving at primary and secondary levels, which are presented as integral elements of all thematic areas of mathematical thinking. Combinatorics, statistics and logics at primary level could be included here as well as their major aim is to develop logical thinking which are essential elements of problem solving.

### 4.1 Theoretical background of problem solving at primary level

We consider a problem all questions and tasks to which there is no clear-cut, instant direct answer. The set of steps leading from the problem itself to its solution is called: logical train of thought. The functions of thinking are constructed from various elements (analysis, synthesis, comparison, addition, abstraction, analogies).

In word problems the essential elements of information are embedded which help the solution of problems instead of „pure" mathematical formulations.

We have to show to the students that mathematics is present everywhere in the world, its knowledge makes our life easier. If we make students able to see mathematics in their own life we foster meaningful and relevant problem solving thinking in them, thus they become ready to learn mathematics. (Rief, 2006)

Word problems are included in education from the first year, in line with the actual content. They are seen as a test measuring whether students acquired the given mathematical model. During the first years of education students learn to solve one problem in one task formulated in a question which enables them to prepare for the real life.

The typology and models of word problems are classified as follows:

– The model of the task can be a number problem, an open sentence or a problem depicted in other way.
– The problems can be classified according to their results: a number, number pair, data – is there a solution, multiple correct answers, no solution to the problem.
– There is a distinction between simple and complex problems. Simple problems can be solved with one step, with one function whereas complex problems require more steps and functions applied.
– The wording of a problem can be „direct" or „reversed" referring to the way of finding the solution.

Well-chosen word problems play a major role in the development of students' problem solving abilities as well as their mathematical skills. Emphasis is put on well-designed textual tasks which depict a real life situation. Although we can call word problem any textual task, not all of them are able to depict real life situations. In the primary level education textual tasks for students follow the actual field of mathematics being taught. Word problems are meant for practicing the routine mechanic operations and mathematical procedures. (Vargáné dr. at all.)

There is a common belief of students that word problems always have a correct solution which could be found with the use of numbers in the text and application of one or two basic functions. The real task is not in modeling a real life situation, but finding the appropriate „genre" of the problem.

What is a realistic word problem? According to Csíkos Cs. realistic word problems are organized by gradual levels and their actual meaning for problem solving:

**1st level:** the things included in textual tasks are derived from the real life surrounding us.

**2nd level:** the tasks are relevant, based on our experiences, not only "cover stories" of a numeric task.

**3rd level:** the application of everyday knowledge is essential.

**4th level:** authentic, nontransparent tasks. (Csíkos, 2008, 2009)

The skills needed for problem solving in word problems are naturally complex, the classification of elements: skills, knowledge and talent needed for problem solving could be even more complex.

The problems meant for students in the first 4 grades belong to 1$^{st}$ and 2$^{nd}$ level, in the consequent years there is a gradual increase towards the 3$^{rd}$ and 4$^{th}$ level.

György Pólya (1962) presented a general problem solving scheme for the mathematical problem solving. There are several phases in the process: the first one is the understanding. By understanding we mean the understanding of the essence of the problem, the problem solver sees what needs to be answered in the given problem.

The second phase is focused on finding relations among data. It is essential for correct problem solving to acquire the relation among data in the problem and the relation among questions to be answered. This phase is given special attention to in Polya's works (Pólya, 2000).

The planning is followed by the implementation. The last phase, considered to be the most important, is the verification. During verification we need to examine if the solution is the answer to our original question indeed, if the outcome satisfies all the criteria set to the solution (Kelemen, 2010). The successful problem solving requires the student to read the text of the problem several times.

While referring to problem solving of word problems at the primary level according to our tradition we apply the above presented Pólya's model.

It is much harder for students to solve textual tasks than arithmetic, counting tasks. (De Corte, 2001; Dobi, 2002).

According to our assumptions the understanding of the problem, its „translation into mathematics" poses difficulties, which means the analysis of the problem presentation.

By problem presentation we mean those elements of problem solving, which are related to mapping, understanding, translation into mathematical operation, construction of a plausible problem solving plan. By construction we mean the performance of operations in arithmetics and algebra. We can conclude that the understanding of the meaning of the problem is the most creative element of solving mathematical textual tasks; the success is mostly dependent on the accuracy of the chosen representation strategy. (Mayer & Hegarty, 1996).

### 4.2 Problem solving at primary level in the innovated national curriculum

Mathematics teaching at primary level is divided into 5 thematic areas. This division remains at later stages as well. Those are the followings:

Numbers, calculations with numbers

Sequences, relations, functions, tables, diagrams

Geometry and measuring

Combinatorics, probabilities, statistics

Logics, reasoning, proofs.

The major aim of teaching mathematics at primary level is the development of skills needed for individual learning and application of knowledge. In order to achieve this goal students acquire experiences leading to cognition related to the age of students. More specific classification of the goals:

– Use of professional language, tables, graphs and diagrams related to the age of students,
– Develop numeric skills of students by numeric calculations done by heart, in writing and using calculator,
– Develop the orientation of students in plane and in space,
– **Develop relations between mathematics and reality by problem solving based on using inductive methods to develop mathematical conceptions, logical and critical thinking,**
– Develop the skills for using IT technologies (calculators, computers), searching, analyzing and saving information,
– Enhance the skills of students to learn individually,
– Strengthen the moral and ethical development of students by the use of systematic, differentiated conduct of work.

Mathematics develops mathematical thinking of students, which is necessary for problem solving in real life situations. Mathematical thinking is developed on a competitive basis that guarantees high level of skills for the application of mathematic knowledge in real life (ŠVP, 2015).

In summary the major aim of primary mathematics teaching is the development of thinking based on understanding, introduction of a two way comprehension between real life situations and mathematical models together with their gradual use (ŠVP, 2015).

The use of acquired mathematical knowledge used outside the schools, individual thinking, development of problem understanding, development of the acquisition of problem solving strategies have become primary goals in education. These goals are not related to merely primary education, but to further education as well. The foundation of the basis is mostly the major task of the primary education.

The book market of Slovakia follows mainly the trends of the national curricula. Schools are obliged to choose from different families of books some of which follow the national curricula closely whereas there are others which do it rather loosely.

Schools with Hungarian as the main language of instruction are not in such a lucky situation. In the past 10 years Hungarian schools in Slovakia could choose from a single family of books which have not always been in line with the new requirements of ŠVP. In the case of teaching problem solving and combinatorics has been even more evident. Teachers have to supplement their own materials to the existing books in order to prepare students for national tests; the missing parts have been mostly connected to probability counting and statistics. The word problems in existing books focus only on the use of routine and mechanic repetition of mathematic procedures, the development of problem solving skills are missing.

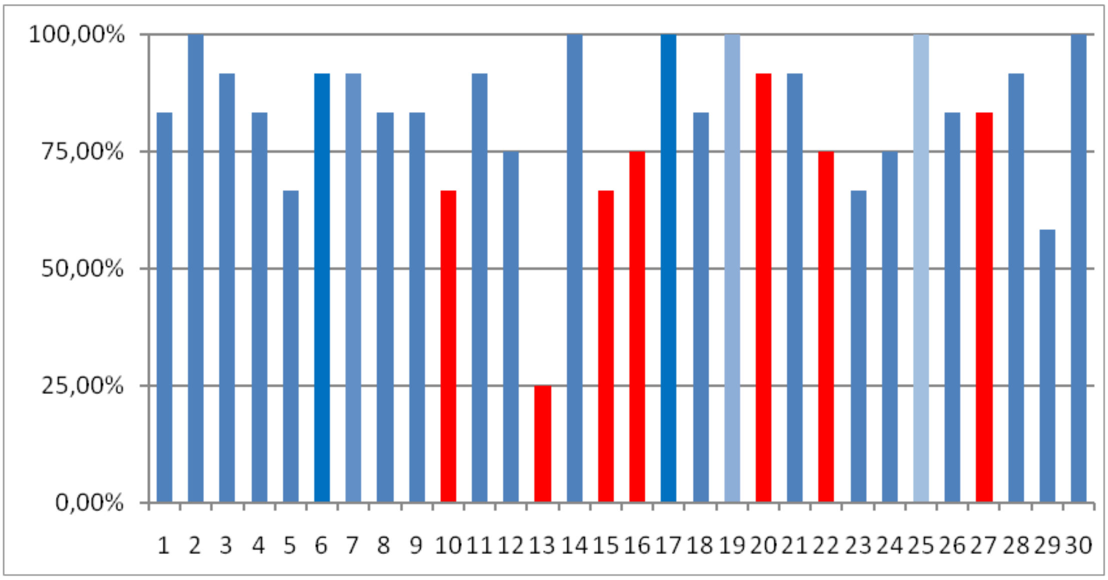
According to the results of the national tests it has been highlighted that the existing books in use need further supplementation with tasks focusing on problem solving and probability counting (Testovanie 5, 2016). Hungarian schools in Slovakia which use elder books need to take this information into account even more intensively.

### 4.3. Success rate of students in problem solving compared to national results

In our work we focused on the problem solving tasks in the fields of Monitor 5 mathematics tests. We put emphasis on tasks including combinatorics, probability counting and statistics. The Monitor 5 test included 18 textual tasks from real life situation, which comprised 60% of the total tasks. These tasks were no. 5, 6, 7, 8, 9, 10,12, 13, 15, 17, 18, 19, 20, 21, 22, 25, 27, 30. We cannot label all of these tasks as real problem solving tasks, as some of them were only texts including routine, mechanic operations. Only 7tasks out of 18 (30%) be regarded as real problem solving tasks, namely 10, 13, 15, 16, 20, 22, 27.

Problem solving tasks were defined as tasks including more than one routine operation line, in which the operation algorithm has to applied not in the regular form, or the texting is different from a routine task.

If we look at the Graph 2 containing the success rate of tasks, we can see the place and success rate of the problem solving tasks compared to the others.




Graph 2: Success rate of tasks involving problem solving (source: own calculation).

The easiest problem-solving task was the task no. 20, the task involved numbers and operations with numbers (success rate was 91,67%). The solution of the task requires the combined use of basic operations in a non-routine situation, the solution could be reached in several steps. Only the weakest student in the group could not solve this problem, this task was included in the easiest problems among 5th grade students as well.

The easy problems included the task no. 27 (success rate 83,33%). The solution of this task required the ability to create a correct sequence – male students had to be put into sequence according to height, based on the given relations. It was regarded as a medium easy task by 5th grade students.

Medium easy tasks in problem solving were tasks no. 16 and 22 (success rate 75%). The task no. 16 included combinatorics and probability counting. Data had to be extracted from graphs and processed by the use of basic operations. This task was regarded as a medium difficult problem among 5th grade students.

The task no. 22 included logics, the effect of the change in initial data had to be estimated by logical thinking, the solutions were provided as multiple choice. The success rate among university students was similar to 5th grade students, the task was regarded as medium easy.

Tasks no. 10 and 15 could be regarded as of middle difficulty (success rate 66,66%). Almost one third of all university students had problems solving this task, which is a large share if we compare their time spent on mathematics learning to that of 5th year elementary school students. These two tasks were regarded as of middle difficulty by 5th year students as well. Both tasks include numbers and operations with numbers, basic operations have to be combined in a non-routine situation in more steps, the text was harder to be understood.

The most difficult task was no. 13 both for university and elementary school students, only 3 students were capable of solving it (success rate 25%). The formulation of the text is ordinary, it is a task in combinatorics: in how many variations can a grandmother divide 3 presents among her 3 grandchildren. It is an alarming fact that elementary school students were more successful than university students in solving this task, although their success rate was the lowest her as well (28,7%).

It is even more disturbing that most of the students, 8 of them tried to find a formula (3X3 or simply answering 3) in their memory. Our university students had one semester mathematics devoted to combinatorics, probability counting and logics in the primary level mathematics teaching.

The fact that these students amount to 2/3 of all Hungarian students majoring teacher training in Slovakia is worrisome for the future of Hungarian schools and community in Slovakia.

Only 3 university students had their success rate above 90% and 3 university students accomplished the national average (60%). If a student scored wrong at any of the problem-solving tasks, at least two other answers to such tasks were wrong, some even failed 5 out of these 7 tasks.

## 5. COMPREHENSIVE ANALYSIS OF THE OUTCOMES OF TESTING – CONCLUSION

The enhanced use of problem solving, combinatorics and geometry has been advised to teachers as requirements in the national curricula, which have been covering a wider spectre of themes than included in the teaching books. These concepts and procedures have to be acquired in order to proceed with advanced content in mathematics teaching in further grades (Testovanie 5, 2016).

The fact that university students have only 20-25% better result than the national average of elementary school students cannot be regarded as satisfactory. The survey clearly points out those fields which have to be strengthened: combinatorics, logics and reasoning, geometry, concepts which have to be elaborated on so that future teachers could use them during their work.

The survey has shown an alarming fact that students are not capable of using basic counting algorithms, and understanding texts posed difficulties for them as well. The success rate (82,5%) might seem satisfactory, but taking into account the fact that the tasks were meant for elementary school students, much better results had been anticipated. There was no student with 100% success rate, whereas there was a large group of students scoring around 60%.

In the future we would like to conclude a survey among bachelor level students and compare it to master level students. The main focus of our future research is on the structure of mathematics teaching in the nursery and teacher training: what is needed for the enhancement of the skills and knowledge of future teachers to make them capable of teaching more successfully at primary levels.

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# CHARACTERISTICS OF A „GOOD PROBLEM"

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*In order to identify and define some of the characteristics that make a „good problem", this paper presents three mathematical problems, and analyses them according to three sets of criteria – stated by Schoenfeld (in press), Friedlander & Arcavi (2017), and Even (1999). As a conclusion, I propose a unified set of characteristics of a „good problem".*

## INTRODUCTION

I would like to consider problem solving as a basic instructional activity that takes up a considerable part of learning mathematics for ***all*** students – rather than its being a „top activity" aimed at „top students". The main purpose of this paper is to identify and discuss some characteristics that are frequently recognized only in an intuitive way as making a „good mathematical problem".

The characteristics of a „good problem" were discussed in several papers and on many occasions. I chose to base this paper on the following three sources:

a) In his draft paper commemorating the untimely passing away of the British master designer Malcolm Swan, Schoenfeld (in press) claimed that a good problem should have the following characteristics:

- relatively accessible – i.e., easily understood and does not require a lot of vocabulary or machinery in order to make progress on it.
- can be solved, or at least approached, in a number of ways –thus allowing students to see that the „bottom line" is not just getting an answer, but seeing connections, and to consider directions or approaches that should be pursued.
- serve as introduction to important mathematical ideas, and illustrate important problem solving strategies, and heuristic skills.
- serve as „seeds" for honest-to-goodness mathematical explorations – i.e., lead students to more problems, and possibly allow them to proceed and make the domain their own.

b) In our book – Friedlander and Arcavi (2017), we present a set of algebra problems, and analyse them according to a list of student competencies. In the context of this paper, a problem that requires and promotes some of the following competencies can be considered a „good problem":

- understanding and applying concepts – allowing students to understand the meaning of concepts and to apply them in a variety of contexts, representations and solution methods.

- global comprehension – requiring students to see an expression, an equation, a function, a graph or a figure as an entity, that can be identified and operated upon as a unit in itself.
- modelling real or mathematical situations, relationships, or properties – promoting students' ability to construct, analyse and operate on mathematical models, interpret the results, and on that basis, derive new knowledge about the situations they represent.
- reverse thinking – requiring students to resort to backward thinking, or to reconstruct a procedure already performed, but missing.
- generating examples – encouraging students to explore, try, create, and review their understandings by generating examples or counterexamples to a given assertion.
- monitoring own or others' work – requiring students to understand, interpret, and evaluate solutions produced by themselves or by others, and to compare results obtained by working with multiple representations of a concept.
- generalizing – requiring students to identify patterns, relations and properties, and express them in various representations.
- justifying and proving – allowing students to engage in proofs of statements or properties either provided by the teacher or conjectured by themselves.

c) In a paper that describes an inservice course for teacher leaders and teacher educators, Even (1999) provides a list of characteristics that formed the basis for a sequence of activities aimed to promote the participants' ability to identify and analyse „good problems". According to her, a good problem should have most of the following attributes:

- provides a meaningful context.
- supports extended discussion and allows an investigation of a big problem.
- allows for multiple correct solutions.
- allows for mathematical connections between representations and domains.
- allows for various levels of presentation and solution are possible.
- presents a genuine need for use of algebra.
- is clearly phrased.
- stimulates thought.
- is interesting.
- causes conflict.
- is not frustrating.

In the following sections, I will analyse three problems that I consider as qualifying to be „good”. Besides my personal classroom experimentations, these problems are presented and analysed in Friedlander and Arcavi (2017). In the final section I will base myself on the characteristics mentioned in this introductory section and on the analysis of the three problems presented and analysed next, to derive and generalize a list of characteristics of a „good problem”.

**PROBLEM #1 – REVERSED NUMBERS**

In this activity (figure 1), students are encouraged to use algebra or other non-symbolic ways to investigate the difference between a two-digit number and its „reverse”. The stages of this activity are as follows:

- working through a class of numerical examples that share a common property (items a, b and c);
- observing and organizing examples, and looking for patterns (items d and e);
- generalizing a pattern (item f);
- providing an explanation for the generalization (item g).

Choose and write down several two-digit numbers whose tens' digit is larger than their units' digit.

a) Write your numbers in reversed order.

b) Find the difference between each number and its reverse.

c) Compare your results with those of your peers.

d) Group your examples according to the results.

e) Can you see a general pattern in your grouping?

f) Use algebra or any other method to justify your conclusions.

Figure 1: The problem of *Reversed Numbers*.

Through their work, students may reach various conclusions about the obtained results:

- they are one of the numbers 9, 18, 27, 36, 45, 54, 63, 72, or 81
- they are always a multiple of 9
- they are both a multiple of 9 and a multiple of the differences of the number's two digits.

Figure 2 presents three student justifications to the observed pattern (item f in the problem).

I would like to mention here three outstanding characteristics of this problem:

a) The problem addresses important mathematical ideas, topics and techniques:

- the idea that contrary to the commonly held belief that proving belongs almost exclusively to the domain of geometry, algebra is also a powerful tool for proving numerical patterns
- the topic of algebraic modelling of the decimal system
- the topic of numerical divisibility
- strategy of employing arithmetical reasoning in generalized situations.

b) The problem allows for both multiple solutions and multiple solution methods:

- more than one number have a given difference of itself and its reverse – for example, the difference of 36 is obtained by 41, 52, 63, 74, 85 and 96
- figure 2 shows that more than one solution method can be employed to justify the emerging pattern.

c) The problem addresses important problem solving strategies and heuristic skills:

- *representing, modelling, and interpreting*. In order to justify algebraically the observed pattern (item g), students are expected to use an algebraic representation of a two-digit number (for example, 10a + b), construct a model for the difference of such a number and its reverse [(10a + b) – (10b + a)], and then interpret the obtained result [9(a – b)].
- *generating examples*. Item a of this task requires reading and understanding the general statement posed verbally („two-digit numbers whose tens' digit is larger than their units' digit"), following instructions („choose and write down several two-digit numbers"), verifying that the selected choice of examples relate to the investigated property, and finally looking for other relevant examples.
- *monitoring own or others' work*. This strategy can be applied throughout the whole solution process.
- *generalizing*. Students are required to notice common regularities, to formulate a hypothesis (each result is a multiple of 9 and the original number's difference of digits), and to express it in a verbal or symbolic form.
- *justifying and proving*. Once a hypothesis was formulated, students are required to address the need for a general argument that goes beyond specific examples, and holds for all the relevant cases. The general justification can be based on symbolic or verbal argumentations.

Solution (a)

Denote the unit digit by x, and the digit-difference by d.

Thus, the two-digit number is represented by $10(x + d) + x$

Its reversed number is $10x + (x + d)$.

Thus, their difference is 9d.

A variation of this method would be to calculate

$10x + y – 10y – x = 9(x – y)$ (where $x > y$).

Solution (b)

This student solution is based on the arithmetic algorithm of vertical subtraction.

$$\begin{array}{rll} - & x + d & x \\ & x & x + d \\ \hline & & \end{array} \;\Rightarrow\; \begin{array}{rll} & \multicolumn{2}{c}{\text{Borrow 1} \longrightarrow} \\ - & x + d & x \\ & x & x + d \\ \hline & & \end{array} \;\Rightarrow\; \begin{array}{rll} - & x + d - 1 & x + 10 \\ & x & x + d \\ \hline & d - 1 & 10 - d \end{array}$$

The value of the difference is $10(d – 1) + (10 – d) = 9d$.

Solution (c)

When we reverse the digits of a two-digit number:

The tens' digit becomes smaller by their difference d, and therefore the reversed number is decreased by 10d.

The units' digit becomes larger by d, and therefore the reversed number is increased by d.

Overall, as compared to the original number, the reversed number is smaller by 9d.

Figure 2: Some solutions to the problem of *Reversed Numbers*.

## PROBLEM #2 – FOLDING A SQUARE

This problem (see figure 3) requires the identification, description, investigation and application of patterns related to the area and the perimeter of the shapes obtained by a repeated folding process that starts with a square.

Students are expected to consider numerical, geometrical, and algebraic aspects of the following patterns:

- The folding creates a sequence of alternating squares and rectangles – decreasing in side length, area, and perimeter.
- All rectangles are similar to a 1x2 rectangle.
- At each step, the area of the shape is halved.
- The ratios between the perimeter of each shape and that of the previous shape in the sequence alternate between $\frac{3}{4}$ (rectangle as compared to a previous square) and $\frac{2}{3}$ (square as compared to a previous rectangle).
- At each second step, the perimeter of each shape is halved.

The process of generalizing the pattern related to the ratio perimeter can follow two routes:

– inductive generalization based on the collection and analysis of data,

– deductive generalization based on a global analysis of the problem situation, and on general reasoning.

In an interview conducted by Friedlander and Arcavi (2005) with two pairs of students (coded by their first names as MS and MG), the interviewers found that contrary to their expectations, both pairs reached, at the initial stage of predictions, generalizations that were „scheduled” to be reached only later on, and on the basis of the collected data. By examining their folded paper square and the drawing of the folding process (see fig. 3), the students considered visual and global aspects regarding the sides that were „lost” through folding, and made the following predictions:

MG: *It [the perimeter] gets smaller by the length of the side that gets halved.*

MS: *In my opinion it [the perimeter] will be 3/4. The vertical lines will stay and the horizontal lines lose one half and one half – and that's a whole side.*

[After the interviewer asks *And what happens from the second to the third shape?*] *It comes out 4/6 because we are left with 4 out of 6 halves* [of the longer sides of the rectangle].

Thus, both pairs produced general patterns at a very early stage of the activity – MG is reasoning additively, by looking at differences, whereas MS is thinking proportionally, by considering ratios.

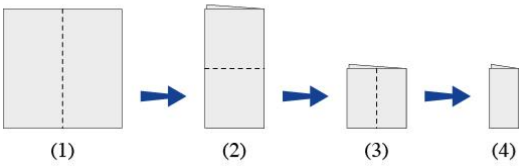
I would like to mention here three outstanding characteristics of this problem:

a) The problem is accessible:
   - the situation of paper folding is familiar to students, and can be easily understood
   - the wording of the tasks does not require a lot of vocabulary or machinery
   - it does not involve many preparatory or supportive tasks.

(1) Take a square sheet of paper.

(2) Fold it along two of its sides.

(3) Fold the obtained rectangle along its two longer sides.




(4) Fold the obtained square again and continue folding the same way.

**Part 1 – Collecting Data**

1. Suppose we start with a square of 8x8 units of length. Find the side-lengths of the folded shapes obtained through the first six steps of folding.

| Step # | Horiz. side | Vert. side |
|---|---|---|
| Start | 8 | 8 |
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |

2. Generalize for four consecutive folding steps that start with a square of *a* by *a* units of length.

| Step # | Horiz. side | Vert. side |
|---|---|---|
| Start | a | a |
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |

**Part 2 – Perimeter**

3. Predict:
   - the ratio between the perimeter of a folded shape and the perimeter of the shape obtained in the previous step;
   - the ratio between the perimeter of a folded shape and the perimeter of the shape that precedes the previous step (namely, two steps backwards).

4. Check your predictions in three different ways: by several examples, by algebra, and by geometry (beware of hasty generalizations).

**Part 3 – Questions**

5. A square of 64x64 is folded repeatedly 12 times, as indicated at the beginning. Is the final shape a square or a rectangle? Find its area and its perimeter.

6. A 5x10 rectangle is obtained after 7 folding steps of a given square. Find the area and the perimeter of the given square.

Figure 3: The problem of *Folding Squares*.

b) The problem provides a meaningful context that has the following attributes:

- mathematically worthwhile – from a numerical, algebraic and geometrical point of view
- authentic – it has the potential to raise students' interest and motivation to discover and justify a quite puzzling pattern.

c) The problem addresses important problem solving strategies and heuristic skills:

- *understanding and applying concepts*. The problem promotes understanding concepts related to area and perimeter (for example, variation patterns, algebraic expressions, sequences, and proportions).
- *representing, modelling, and interpreting*. Students are required to construct algebraic models that reflect a process of repeated folding of a square.
- *reverse thinking*. The last question (item 6) requires students to apply the patterns found in a reversed direction – namely, following an „unfolding" process.
- *monitoring own and others' work*. An analysis of the same process by numbers, symbols, and geometrical shapes provides opportunities to compare findings. The required predictions also allow one to reflect on the expected results, and to compare the initial expectations and the actual findings.
- *generalizing*. Step-by-step calculations are preceded by predictions of general patterns, which are followed by geometrical and quantitative reasoning and algebraic generalizations. Patterns of change are generalized and expressed in algebraic expressions and in words.
- *justifying and proving*. Patterns of change are justified by using algebraic and geometrical considerations, and by proportional reasoning.

## PROBLEM #3 – INTEGRATING AREAS

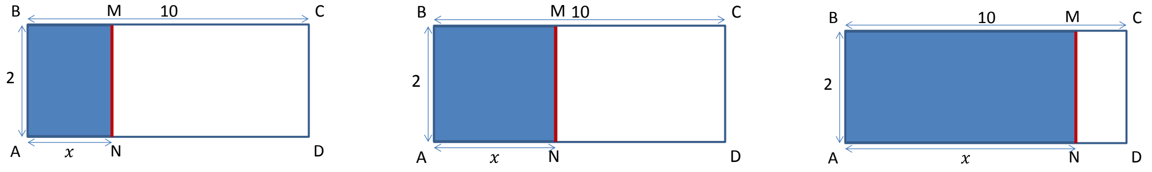
This problem (see figure 4) is based on Arcavi (2008) and on Friedlander and Arcavi (2017), and it engages students in:

- creating, interpreting, and exploring symbolic and graphical models of geometrical phenomena related to areas;
- making transitions between geometrical situations and their symbolic and graphical representations;
- analysing functions at a global level (e.g. linear/quadratic functions, increasing/decreasing functions, and the sum of functions);
- analysing functions at a local level (e.g. finding specific values).

The problem can be also considered as an intuitive introduction to the topic of integration to be studied in calculus courses.

Part 1

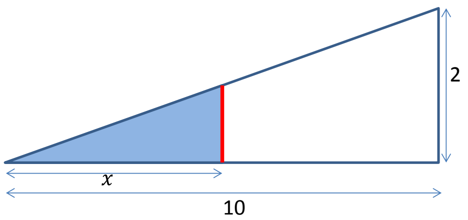
Segment MN (parallel to AB) moves from left to right in rectangle ABCD.
As shown in the figure below, increasing the length of segment AN ($x$) increases the area of rectangle ABMN as well




1.a. What is the area of the shaded rectangle if $x = 3.5$ cm?
 b. What is the value of $x$, if the area of the shaded rectangle is 19 $cm^2$?
2. Write an algebraic expression for the function $f(x)$ that represents the area of the shaded rectangle for increasing values of $x$.
3. Define the domain and the range of the area function $f(x)$, and sketch its graph.

Part 2

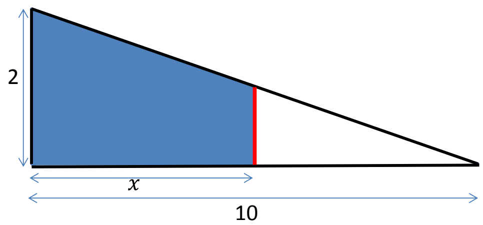
In this part we explore a similar situation, but this time the moving line slides within a right triangle, as shown in the figure on the right.




4. Write an algebraic expression for the area function $g(x)$ that represents the area of the shaded triangle for increasing values of $x$.
5. Define the domain and the range of the function, and sketch its graph.
6. Find the value of $x$ for which the area of the shaded inner triangle is half the area of the large triangle.

Part 3

In this part, we explore the function that represents the variation of the shaded area, as shown in the figure at the right.




Write an algebraic expression for a function $h(x)$ that represents the area of the shaded trapezoid for increasing values of $x$.
Define the domain and the range of the area function $h(x)$, and sketch its graph.
9. Dylan claimed that the area function $h(x)$ is decreasing. Do you agree? Explain.
10. Find the value of $x$ for which the area of the shaded inner trapezoid is half the area of that of the large triangle.
11. Show in a geometrical and an algebraic way that $f(x) = g(x) + h(x)$.

Figure 4: The problem of *Integrating Areas*.

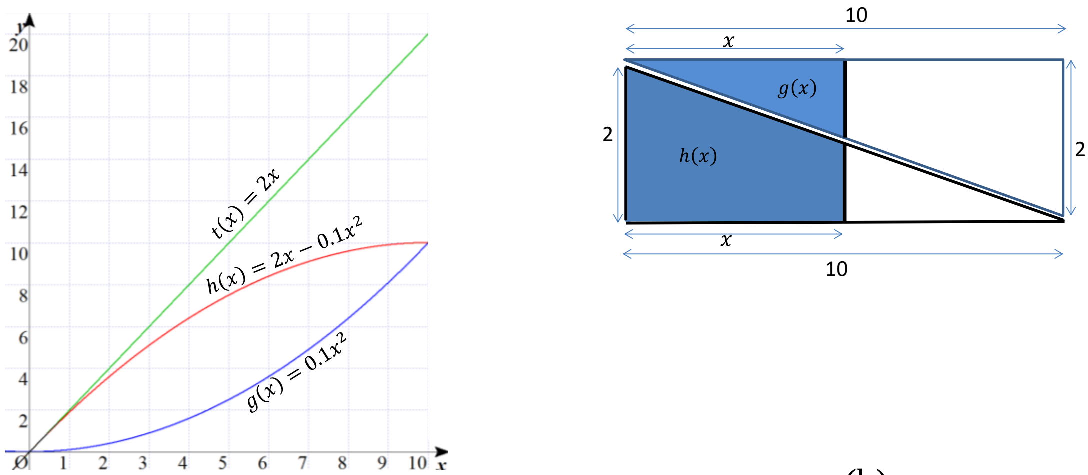
I would like to mention here two outstanding characteristics of this problem.

a) The problem provides seeds for mathematical explorations:

As indicated in Arcavi (2008) discrepancies were observed between students' expectations or predictions of the corresponding area functions and graphs and the mathematically obtained results. For example, in some cases the graph of the cumulative area function for the rectangle in part 1 was predicted to be that of a constant function (rather than an increasing linear function), whereas the graph of the cumulative area function of the right triangle with a „negative-slopped" hypotenuse in part 3 was predicted to be that of a decreasing function (rather than an increasing quadratic function). This cognitive dissonance raised students' curiosity, and encouraged them and their teachers to investigate area variations in additional shapes, and finally provided a meaningful context for exploring the cumulative area function below the graph of a function (rather than that of a geometrical shape).

b) The problem allows for connections:

The problem requires systematic transitions between geometric, symbolic and graphical representations of the given situations. For example, in the last item of the problem $g(x) + h(x) = f(x)$ can be justified by using various representations. The algebraic procedure provides a surprising result, namely, that $g(x) + h(x) = \frac{1}{10}x^2 + 2x - \frac{1}{10}x^2 = 2x = f(x)$. As shown in figure 5(a), performing the addition graphically is equally surprising – the point-by-point addition of the y-coordinates of two quadratic functions yields a linear function, but it is when we turn to the geometrical phenomenon that these surprises can be easily explained. As shown in figure 5(b), joining the area of the shaded triangle, represented by $g(x)$, with the area of the trapezoid, represented by $h(x)$, results in the area of the rectangle represented by $f(x)$.




(a) (b)

Figure 5: Showing that $g(x) + h(x) = f(x)$ in graphical (a) and geometrical (b) representations.

Besides the aspect of multiple representations, the problem requires students to integrate geometry and algebra and furthermore it provides an intuitive introduction to the topic of integration to be studied in calculus courses.

## CONCLUSION

As proposed in the introductory section a conclusion, I will attempt to derive a unified list of characteristics of a „good problem” – based on three sources – Schoenfeld (in press), Friedlander and Arcavi (in press), and Even (1999), and on the three problems presented and analysed in the previous sections.

Note that in order to avoid repetitions, the analysis of each of the three problems mentioned only several of its more outstanding characteristics. The following list was aimed to reflect the spirit of the three sources, but is not intended to be comprehensive or exhaustive.

a) **Structure:** The problems are **accessible**. They are based on easily understandable situations (for example, paper folding, experimenting with whole numbers, a segment moving on the top of a simple geometrical shape), their wording does not require a lot of vocabulary, and they do not involve many preparatory or supportive tasks. However, most of them are **big problems** that allow for extended investigations and discussions of a problem situation.

b) **Content:** The problems are based on **meaningful contexts** that promote **important mathematical ideas** (for example, number patterns, area and perimeter variation and variation of ratios) and provide „seeds to honest-to-goodness explorations” (Shoenfeld, in press).

c) **Cognition:** The problems are **thought provoking** – i.e., their solutions involve thinking processes and problem solving strategies – such as applying learned concepts, global comprehension, modelling, reverse thinking, generating examples or counterexamples, monitoring work, generalizing and justifying.

d) **Connections:** The problems allow for connections **between representations** (numerical, symbolic, graphical, geometrical, verbal), **between solution methods** (see for example, solutions of Problem #1 presented in figure 1, and solutions of Problem #3 presented in figure 5) and **between mathematical domains** (for example, between arithmetic and algebra in Problem #1, between arithmetic, algebra and geometry in Problem #2, and between algebra, geometry and calculus in Problem #3).

e) **Affect:** Meaningful contexts and extended investigations have the potential to present a genuine need for the use of mathematics, and hence to raise **student curiosity, motivation and interest**.

As a last comment, I would like to note that the quality of a mathematical problem (as discussed in this paper) is not the single variable in our attempts to ensure an effective learning environment. Its classroom implementation in general and the accompanying student-student and student-teacher interactions in particular, are crucial as well. In other words, task design and classroom pedagogy influence each other: higher-quality problems may compensate to a certain degree for lower-quality interactions, and conversely, higher-quality interactions may compensate to a certain degree for lower-quality problems.

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# PROBLEMS IN THE CONTEXT OF THE THALES CIRCEL

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*We start with a segment $\overline{AB}$ and a circle which has $\overline{AB}$ as a diagonal. If we then choose a point C of this circle with C ≠ A, B and move C along the circle we always get a right-angled triangle. This is the statement of the well-known theorem of Thales.*

*In connection with this theorem we get some interesting problems if we look out for other measures and special points of these right-angled triangles. Which traces are built by the midpoint of the inner circles or the gravity center? How does the area of these triangles or the length of its sides change? How does the square determined by A, B, C change?*

*Such activities encourage the students to see a well-known theorem in a broader context and to look out for possible problem posing in a given problem field. Besides, all these questions can be supported very well in respect to finding problems by a Dynamic Geometry Software. Moreover students can learn that some classical problems also can be seen in a dynamic way.*

## INITIAL POSITION

Let be given a segment $\overline{AB}$, a circle which has $\overline{AB}$ as a diagonal and a point $C$ on the circle with $C \neq A, B$. The midpoint of $\overline{AB}$ we call $O$ and for the sides of the triangle $ABC$ we use the following usual notations $|\overline{BC}| = a$, $|\overline{AC}| = b$, $|\overline{AB}| = c$ as well as $h$ for the height through $C$. Furthermore for the angle measures we will use the following naming: $|\angle CAB| = \alpha$, $|\angle ABC| = \beta$, $|\angle BCA| = \gamma$ and $|\angle BOC| = \varphi$.

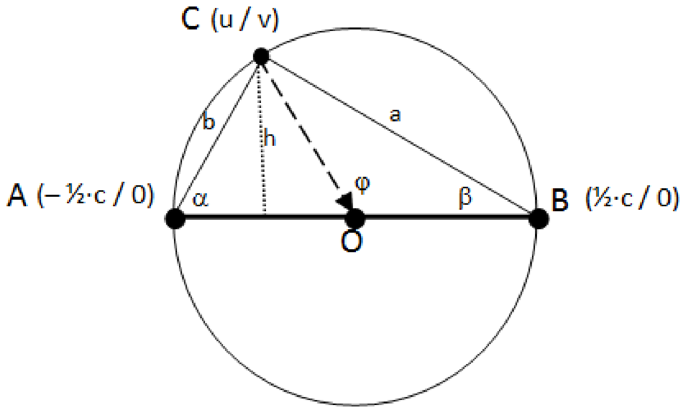


Figure 1

Because of the Theorem of the Thales Circle we get

(I) $\gamma = 90°$ and with help of the *theorem of the sum of the angle measures* also $(\alpha + \beta) = 90°$.

For an ***analytic description*** of this given figure we use $O$ as origin of the coordinate system and $A$, $B$ on the $x$-axis symmetric to $O$ so that $A = (-\frac{1}{2}\cdot c\ /\ 0)$ and $B = (\frac{1}{2}\cdot c\ /\ 0)$. The coordinates of $C$ we name $(u/v)$.

Because $C$ does lie on the Thales Circle of $\overline{AB}$ with help of the *theorem of Pythagoras* resp. trigonometry we get the statement

(II) $u^2 + v^2 = (\frac{1}{2}\cdot c)^2$ or $u = \frac{1}{2}\cdot c\cdot\cos\varphi$ and $v = \frac{1}{2}\cdot c\cdot\sin\varphi$.

Because the triangles $AOC$ and $BOC$ are isosceles triangles by *Thales'theorem of isosceles triangles*we get $|\angle ACO| = \alpha$ and $|\angle OCB| = \beta$. By using the *theorem of thesum of the angle measures* we then easily can find

(III) $\varphi = 2\cdot\alpha$ or $\varphi = 180° - 2\cdot\beta$.

For our ***dynamic view on this figure*** we only want to have a look at the upper half-circle (i.e. $v \geq 0$). As direction of movement of the point $C$ on the Thales circle we will use the direction from „the left to the right", i.e. for a parameter $u$ resp. $\varphi$ resp. $\beta$ this means

$u$: $-\frac{1}{2}\cdot c \to +\frac{1}{2}\cdot c$ and $\varphi$: $180° \to 0°$ and $\beta$: $0° \to 90°$.

## TRACES OF SPECIAL POINTS OF THE TRIANGLE ABC BY MOVING C ON THE THALES CIRCEL

Normally in grade 7 or 8 the students deal with different theorems concerning a triangle. In this time the points of intersection of some auxiliary lines are important. Therefore it is obvious that one has a look out for the movement of such a point while $C$ is moving on the Thales circle.

The ***point of intersection of the perpendicular bisectors*** – the midpoint of the circumcircle – of course is the point $O$. The trace therefore is only one point.

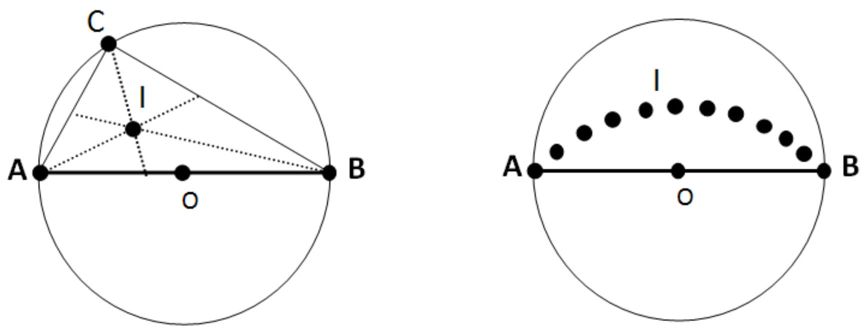


Figure 2

For the ***point of intersection of the angle bisectors*** – the midpoint of the inner circle, named $I$ – we have $|\angle BAI| = \frac{1}{2}\cdot\alpha$ and $|\angle IBA| = \frac{1}{2}\cdot\beta$. The sum of both therefore is $\frac{1}{2}\cdot(\alpha + \beta) = \frac{1}{2}\cdot 90° = 45°$. This results that the angle measure at $I$ for all positions of $C$ on the Thales circle is constant, i.e. $|\angle AIB| = 180° - \frac{1}{2}\cdot(\alpha + \beta) = 180° - 45° = 135°$. Because of the *theorem of the peripheral angle* we get that the trace of I is an arc of a circle which moves from $A$ to $B$ with maximum on the $y$-axis equal to $\frac{1}{2} \cdot c \cdot \tan(22{,}5°) = \frac{1}{2} \cdot c \cdot (\sqrt{2} - 1) \approx 0{,}207 \cdot c$ (see Figure 2).

The ***point of intersection of the heights*** is because of $\gamma = 90°$ the point $C$. Thus its trace is the Thales circle.

The ***point of intersection of the median line*** – the gravity center, named $S$ – lies on the line $CO$. Because of the *theorem about the gravity center* the point $S$ divides the segment $\overline{CO}$ with ratio 2:1. Therefore $|OS| = \frac{1}{3} \cdot |OC| = \frac{1}{3} \cdot (\frac{1}{2} \cdot c) = \frac{1}{6} \cdot c$. Thus $S$ does move on a circle with midpoint $O$ and radius $\frac{1}{6} \cdot c$ (see Figure 3).

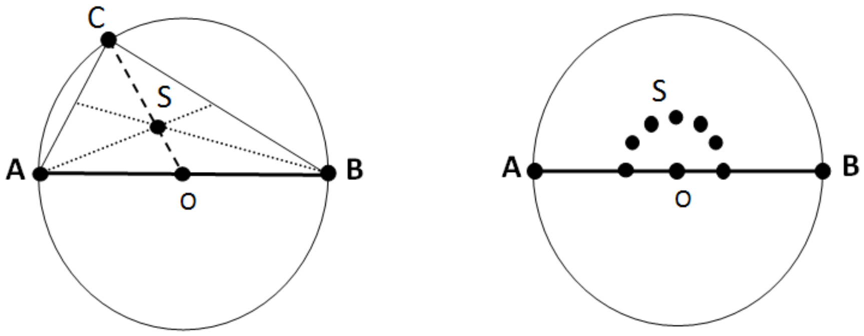


Figure 3

## GRAPHS OF THE HEIGHT AND AREA OF THE TRIANGLE *ABC*

While moving $C$ on the Thales circle we always get right angled triangles; but its shape is changing. A first question therefore may be: How is the area of $ABC$ changing. Because the base $\overline{AB}$ is fixed the area depends only on the height $h$. Thus we start to investigate h and immediately find:

$h = v$ and together with (II) we get $h(u) = \sqrt{\frac{c^2}{4} - u^2}$.

That means – which can be taken also directly – that the graph of $h$ is equal to the Thales circle.

With (IV) and the formula for the area of a triangle we then get:

(IV) The area of $ABC$ is given by $\mathcal{A}(u) = \frac{1}{2} \cdot c \cdot \sqrt{\frac{c^2}{4} - u^2}$.

With an algebraic equalization this results in $\mathcal{A}(u)^2/(\frac{c^2}{4} \cdot \frac{c^2}{4}) + u^2/(\frac{c}{2} \cdot \frac{c}{2}) = 1$ which describes an ellipsewith midpoint $O$ and axis length $\frac{c}{2}$ and $\frac{c^2}{4}$, so that it goes through $A$ and $B$ with maximum $\frac{c^2}{4}$ on the $y$-axis (see Figure 4).

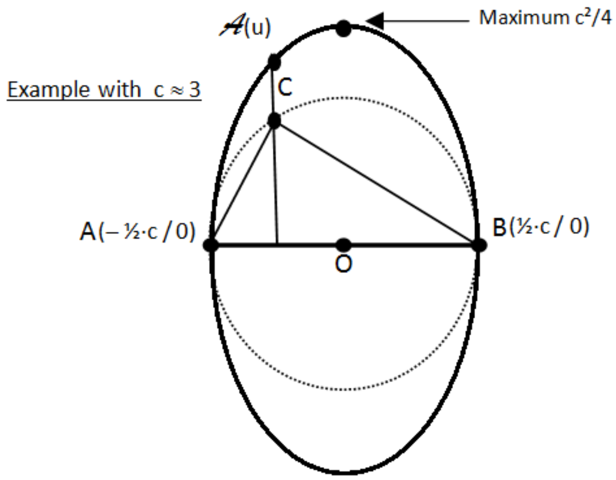


Figure 4

The ellipse lies outside of the Thales circle if $c > 2$ and inside if $c < 2$. For $c = 2$ it is identical with the Thales circle.

## CHANGES IN THE SIDES LENGTHS

Becausec is constant and $b(u) = a(-u)$ we only will have a look at $a$. Using the *theorem of Pythagoras* (or the formula of the distance of two points in a coordinate system) we get $a^2 = (\frac{1}{2}\cdot c - u)^2 + v^2 = \frac{1}{4}\cdot c^2 - c\cdot u + u^2 + v^2 = \frac{1}{2}\cdot c^2 - c\cdot u$ [cf. (II)] and

$a = \sqrt{\frac{c^2}{2} - c\cdot u}$ or $a^2/(\frac{c^2}{2}) + u/(\frac{c}{2}) = 1$

Therefore the length $a$ changes from $c$ [for $u = -\frac{1}{2}\cdot c$] to 0 [for $u = \frac{1}{2}\cdot c$] along an arc of a parabola.

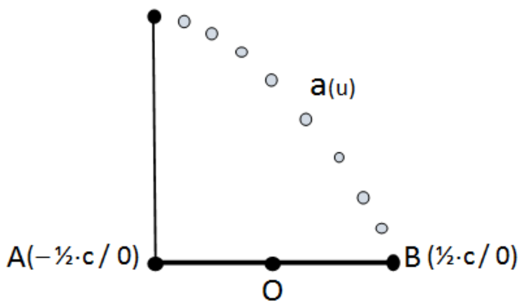


Figure 5

## TRACES OF THE VERTICES OF A RECTANGLE WHICH COVERS THE TRIANGLE *ABC* AND HAS *C* AS ONE VERTEX

Sometimes in school the reverse theorem of the Thales circle is illustrated with a rectangular sheet which is put between two fixed points $A$, $B$ and moved.

Because of the inverse theorem of Thales the upper vertex $C$ then is moving on a circle, the Thales circle.

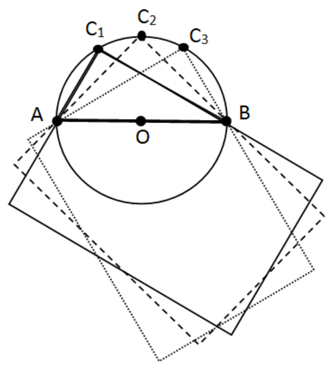


Figure 6

My colleague AlfstNickelsen once asked me if I know the traces of the other vertices of such a rectangle if $C$ moves on the Thales circle. Because I never had heard of an answer to such a question I tried to find the solution by myself.

For this I first took the nomenclature asdone above in the initial position, butfor a little simplification I used $c = 2$, so that $|OA| = |OB| = |OC| = 1$.

With (II) we get $u^2 + v^2 = 1$ and $u = \cos\varphi$ as well as $v = \sin\varphi$.

The other vertices of the rectangle I named $R$, $S$, $T$ (in the direction from $B$ to $A$).

Then I moved the rectangle with $\overrightarrow{CO}$and named the picture points $R^*$, $S^*$, $T^*$.

The picture point $C^*$ is $O$.

If I now move $C$ on the Thales circle this implies a rotation of $OR^*S^*T^*$ around $O$.

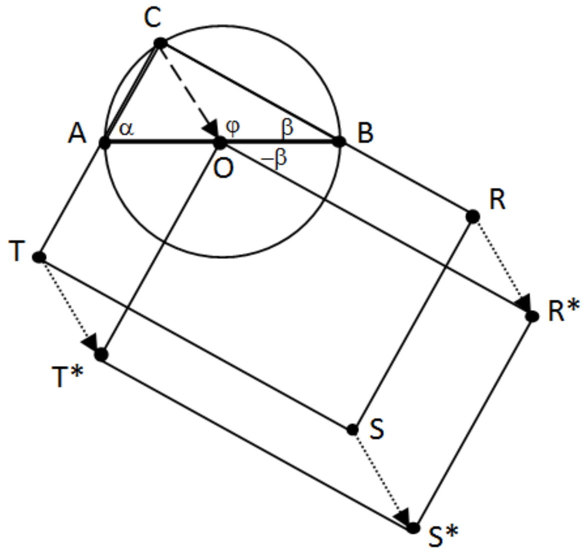


Figure 7

Because of the *theorem of the alternateangle* we have $|\angle BOR^*| = \beta$ but with opposite direction. Thus the movementof $C$ from $A$ to $B$ corresponds to $|\angle BOR^*|$ from 0° to –90°.

Because $|OR^*|$, $|OS^*|$, $|OT^*|$ are fixed sizes we get for the traces of $R^*$, $S^*$, $T^*$ circles with midpoint $O$.Corresponding to $\beta$: $0° \to 90°$ we get $|\angle BOR^*|$: $0° \to -90°$, $|BOS^*|$: $(-\sigma) \to (-\sigma - 90°)$ with $\sigma$ as fixed measure $|\angle S^*OR^*|$ and $|\angle BOT^*|$: $-90° \to -180°$ because of $|\angle BOT^*| = -\beta - 90°$ and $|BOS^*| = -\beta - \sigma$.

With these results and the description of coordinative points in polar coordinates as well as the abreviations$r = |\overrightarrow{CR}| = |\overrightarrow{OR^*}|$we get for ***R*** the following:

$$\overrightarrow{OR} = \overrightarrow{OR^*} + \overrightarrow{OC} = (r\cdot\cos(-\beta);\ r\cdot\sin(-\beta)) + (u / v) = (r\cdot\cos\beta + \cos\varphi;\ -r\cdot\sin\beta + \sin\varphi) \text{ or}$$

$$\overrightarrow{OR} = (r\cdot\cos\beta + \cos(180° - 2\cdot\beta);\ -r\cdot\sin\beta + \sin(180° - 2\cdot\beta)) \text{ by using (III).}$$

Because of $\sin(180° - 2\cdot\beta) = \sin 2\cdot\beta$ and $\cos(180° - 2\cdot\beta) = -\cos 2\cdot\beta$ as well as the formulas for $\sin 2\cdot\beta$ and $\sin 2\cdot\beta$ we finally get

$$\boxed{\overrightarrow{OR} = (r\cdot\cos\beta + \sin^2\beta - \cos^2\beta;\ -r\cdot\sin\beta + 2\cdot\sin\beta\cdot\cos\beta).}$$

To get an idea what this representation of the trace of $R$ by the parameter $\beta$ does look like without a parameter we will use the abbreviation $\overrightarrow{OR} = (x\ ;\ y)$, i.e.

$x = r\cdot\cos\beta + \sin^2\beta - \cos^2\beta$ and $y = -r\cdot\sin\beta + 2\cdot\sin\beta\cdot\cos\beta$.

Replacing then ($\sin^2\beta$) by ($1 - \cos^2\beta$) in the equation of $x$ we can get a formula for $\cos\beta$ in dependence only of $r$ and $x$.

$x = r\cdot\cos\beta + 1 - 2\cdot\cos^2\beta \Leftrightarrow \cos^2\beta - \frac{r}{2}\cdot\cos\beta = \frac{1}{2} - \frac{x}{2} \Leftrightarrow (\cos\beta - \frac{r}{4})^2 = \frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}$ and thus $\cos\beta = \frac{r}{4} - \sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}$.

This expression alsways is defined unique because of $0° \le \beta \le 90°$ we have

$(r-1) \ge x \ge 1$ resp. $-\frac{r-1}{2} \le -\frac{x}{2} \le -\frac{1}{2}$ and thus

$0 \le (\frac{1}{2} + \frac{r^2}{16} - \frac{r-1}{2}) = (\frac{r}{4} - 1)^2 \le \frac{1}{2} + \frac{r^2}{16} - \frac{x}{2} \le \frac{r^2}{16}$.

With $\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2} \le \frac{r^2}{16}$ it follows $0 \le \frac{r}{4} - \sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}$ and with

$(\frac{r}{4} - 1)^2 \le \frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}$ we have together $0 \le \frac{r}{4} - \sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}} \le 1$.

Using the formula $\sin\beta = \sqrt{1 - \cos^2\beta}$ we then get

$\sin\beta = \sqrt{1 - (\frac{r}{4} - \sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}})^2} = \sqrt{1 - (\frac{r^2}{16} - \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}} + (\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}))}$, i.e.

**$\sin\beta$** $= \sqrt{-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2} + \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}}$.

We now insert the above expressions of $\cos\beta$ and $\sin\beta$ in the equation of $y$ and get a functional equation of $y$ with the variable $x$ and the constant $r$.

$$y = -[\frac{r}{2}\cdot(\sqrt{-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2} + \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}}) + 2\cdot(\sqrt{-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2} + \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}})\cdot(\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}})].$$

This still does not give us an idea of the type of the function of trace of $R$. Therefore we take some simple algebraic transformations like the followings:

$$y^2 = \frac{r^2}{4}\cdot(-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2} + \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}) +$$

$$+ 2r\cdot(\sqrt{-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2} + \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}})\cdot(\sqrt{-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2} + \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}})\cdot(\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}}) +$$

$$+ 4\cdot(-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2} + \frac{r}{2}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}})\cdot(\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}) =$$

$$= \frac{r^2}{4}\cdot(-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2}) + \frac{r^3}{8}\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}} +$$

$$+ 2r\cdot(-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2})\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}} + r^2\cdot(\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}) +$$

$$+ 4\cdot(-\frac{r^2}{8} + \frac{x}{2} + \frac{1}{2})\cdot(\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}) + 2r\cdot(\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2})\cdot\sqrt{\frac{1}{2} + \frac{r^2}{16} - \frac{x}{2}} =$$

$$= (-\frac{r^4}{32} + \frac{r^2}{8}\cdot x + \frac{r^2}{8}) + (\frac{r^2}{2} + \frac{r^4}{16} - \frac{r^2}{2}\cdot x) + (-\frac{r^2}{4} + (1 + x))\cdot((1 - x) + \frac{r^2}{8}) +$$

$+\left(\frac{r^3}{8}-\frac{r^3}{4}+r\cdot x+2r+\frac{r^3}{8}-r\cdot x\right)\cdot\sqrt{\frac{1}{2}+\frac{r^2}{16}-\frac{x}{2}}=$

$=\left(-\frac{r^4}{32}+\frac{r^2}{8}\cdot x+\frac{r^2}{8}\right)+\left(\frac{r^2}{2}+\frac{r^4}{16}-\frac{r^2}{2}\cdot x\right)+\left(-\frac{r^2}{4}+\frac{r^2}{4}\cdot x-\frac{r^4}{32}+1-x^2+\frac{r^2}{8}+\frac{r^2}{8}\cdot x\right)+$

$+2r\cdot\sqrt{\frac{1}{2}+\frac{r^2}{16}-\frac{x}{2}}=$

$=1+\frac{r^2}{2}-x^2+2r\cdot\sqrt{\frac{1}{2}+\frac{r^2}{16}-\frac{x}{2}}.$

By bringing only the root on one side of the equation we have

$y^2+x^2-\left(1+\frac{r^2}{2}\right)=2r\cdot\sqrt{\frac{1}{2}+\frac{r^2}{16}-\frac{x}{2}}$ and then get

$(y^2+x^2)^2-2\cdot(y^2+x^2)\cdot\left(1+\frac{r^2}{2}\right)+\left(1+\frac{r^2}{2}\right)^2=4r^2\cdot\left(\frac{1}{2}+\frac{r^2}{16}-\frac{x}{2}\right)$ resp.

$y^4+2x^2y^2+x^4-(2+r^2)\cdot y^2-(2+r^2)\cdot x^2+1+r^2+\frac{r^4}{4}=2r^2+\frac{r^4}{4}-2r^2\cdot x.$

By this we can see that for ***R* = (*x*/*y*)** the following ***equation of grade 4*** is valid.

| |
|---|
| $y^4-(2+r^2)\cdot y^2+2x^2y^2+x^4-(2+r^2)\cdot x^2+2r^2\cdot x+(1-r^2)=0$ |

For ***T*** and the abbreviation $t=|\overrightarrow{CT}|=|\overrightarrow{OT^*}|$ we get

$\overrightarrow{OT}=(-t\cdot\sin\beta+\sin^2\beta-\cos^2\beta;\ \ -t\cdot\cos\beta+2\cdot\sin\beta\cdot\cos\beta).$

With some transformation similar to the above one we get for the trace of *T* also an ***equation of grade 4***.

For ***S*** and $s=|\overrightarrow{CS}|=|\overrightarrow{OS^*}|$ as well as the equation $s^2=r^2+t^2$ (because of the theorem of Pythagoras) we get

| |
|---|
| $\overrightarrow{OS}=(r\cdot\cos\beta-t\cdot\sin\beta+\sin^2\beta-\cos^2\beta;\ -r\cdot\sin\beta-t\cdot\cos\beta+2\cdot\sin\beta\cdot\cos\beta)$ |

With some more difficult transformations then for the trace of ***S*** we get an ***equation of grade 16***.

I think this is the reason that the trace of *R*, *S*, *T* you don't find in a school book.

<u>Remark</u>: Without trigonometry one also can get these equations by using formulas of analytic geometry. For example for *R* we can use the distance formula for |*OC*|, the equality of the slope of *CB*, the slope of *BR* and the distance for |*CR*|:

$u^2+v^2=1$ resp. $v^2=1-u^2$ or $v=\sqrt{1-u^2}$

$(y-0)/(x-1)=(v-0)/(u-1)$

$(y-v)^2+(x-u)^2=r^2$ resp. $y^2-2vy+x^2-2ux=r^2-(v^2+u^2)$

With several transformations we then also get the above functional equation of grade 4 for the trace of *R*. And in a similar way we also get the functional equation of grade

4 of *T* and with a little bit more effort we also get the functional equation of grade 16 of *S*.

## VARIATIONS OF THE ABOVE PROBLEMS

For students that got delighted with some of the above treated problems one can easily expand their inset at the given initial figure e.g. by looking out for other special points of the triangle *ABC* like the vertices and midpoints of the squares above the sides (as used for the theorem of Pythagoras) or the Fermi-point or the midpoint of the Feuerbach circle. Or they can investigate the change of the length of the angle bisector or the median lines.

Other variations concern the initial track of *C* like having still two fixed points *A* and *B* one can move *C* instead on the Thales circle on an arc of a circle between *A* and *B*. We then have also a constant angle measure at *C* but not equal to 90°. Or we can move *C* on an ellipse or parabola. Interesting – and not so difficult – is the variation if we move *C* on a straight line. For example if this line is parallel to the *x*-axis then all triangles *ABC* have a constant area and if *C* moves on a straight line going through *O* then e.g. the point of gravity also moves on this line and the midpoint of the circumcircle moves on the *y*-axis. Or if the trace of *C* is the *y*-axis then all triangles *ABC* are isosceles triangles.

## FINAL REMARK

The theorem of the Thales circle is a theme which belongs to normal geometry education. But a good education should show that almost always one can find new interesting aspects by looking at that theme from another viewpoint or by varying the giving problem. With this the students get a broader view of mathematics and learn posing problems by themselves.

For this it is not important which other aspects or variations are used.But the students have to be active by exploring a problem field and the teacher must give freedom of action to the students – not stick to a narrow fixed curriculum. Many aspects of a normal curriculum then will be trained by the way. Problem solving in this way is important for an evidence-based teaching (see e.g. Hattie 2009 and & Bernhard 2014 as well as Polya1949) and also required in ministerial educational standards (see e.g. KMK 2003 and 2004).

The presentation here is intended to give only stimulation.The above examples of course are not – and do not want to be – an accurate preparation for class instruction. They have to be adapted to the concrete situation in school. Concerning class level I would say that the named themes are possible to treat in several grades above of grade 8 – some questions can be treated already in grade 8 but most of them in grade 9 and some of them only in grade 10 or above.

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# AUGMENTED REALITY IN MATHEMATICS EDUCATION IN PRIMARY

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*Augmented Reality technology enables us to develop new learning methods for mathematics education both in the elementary and primary school. This paper shows the using of Augmented Reality (AR) in educational environments, especially in the mathematics problem solving. In the article, a list of the best AR applications in math learning are presented which can be used for problem solving in constructionist environment. One of them, the Augmented polyhedrons, which was applied in the classroom, is described thoroughly and supported with a qualitative assessment as well.*

## INTRODUCTION

The introduction of problem solving thinking is greatly enhanced by the study of geometric shapes written by Pólya György (1996). Studying the evolution of teaching mathematics, especially geometry with particular attention to the various illustrative modes, we find that the development keeps up with the age we live in. From the teaching of geometry with different shapes of wood, through the method of 'the student must bring any kind of object resembling to a geometric shape from home for tomorrow' to the introduction of computer in math class in some cases, we can meet a lot of methods that were implemented during the decades. The 21st century has brought a totally new opportunity, namely the study of geometric shapes through the extension of reality with the camera of our mobile device. We can observe the 3-dimensional shapes from different angles so that they are floating over our desk, complementing the elements of real space.

Visual language is more capable of spreading knowledge than any other communication device. We can create the attraction of pupils to mathematics by visual representation (AR FlashCards Addition application), because we know that visual impressions remain at a higher percentage than text or verbal information. According to Piaget's research (1978, 1999), the evolution of thinking is manifested simultaneously in several areas: speech, music and expression, shape, form and field of vision, ability to represent a drawing, symbolic communication skills, etc.

Facts and ideas can be transmitted more and much deeper by the visual language than by any other communication tool (Kepes, 1979). Arnheim emphasized the unity of perception and abstract thinking, according to which mental processes do not only consist of operations with words and numbers, but also imaginary thinking.

## AUGMENTED REALITY IN EDUCATION

The term Augmented Reality (AR) was born at Boeing in 1990 by researcher Tom Caudell, and it translates the integration of virtual images in the real world, i.e. the reality is augmented of virtual elements. The integration of such images is made by the use of technology through a mobile device with a camera (computer, tablet, mobile phone) which allows the access to the available contents with AR. One characteristic offered by AR applications is the integration and interaction between the real and the virtual, allowing an enormous versatility and creativity in applications (Coimbra, 2015).

We are talking about virtual reality if the 3D image seen through the Oculus Rift or other spectacles has nothing to do with the view in front of me in reality. We enter a virtual space, where we can turn around, move, and we experience that we have stepped into a totally new world.

In contrast to this, in augmented reality we stay in the same space where elements are projected. We think that augmented reality is more tangible than virtual reality. There is a part which can be perceived in reality, which is expanded with electronic information, such as 3D images, animation, video etc. We therefore create a bridge between reality and virtual reality with which we can exercise great influence on every generation and achieve extraordinary results.

We are talking about the usage of augmented reality in education when with the aid of an application on a mobile device, virtual objects appear in the space and this information helps pupils understand the new material, complements teachers' work, supports self-checking during problem solving and improves spatial abilities. Further benefits were identified and described in Benefits of Augmented Reality in Education Environments (Diegman, 2015) research. The steps of the research were the following: identified relevant publication, analysed the identified publication by coding and grouping benefits.

(25 articles, 6 benefit groups, 14 different benefits and 5 directions)

| | | **Discovery-based Learning** | **Objects Modeling** | **AR Books** | **Skills Training** | **AR Gaming** | **Total** |
|---|---|---|---|---|---|---|---|
| **State of Mind** | *Motivation* | 7 | 4 | 2 | 1 | 1 | 15 |
| | *Attention* | 2 | 0 | 1 | 0 | 0 | 3 |
| | *Concentration* | 2 | 0 | 0 | 0 | 1 | 3 |
| | *Satisfaction* | 1 | 2 | 0 | 1 | 1 | 5 |
| **Teaching Concepts** | *Student-centered Learning* | 2 | 0 | 1 | 0 | 0 | 3 |
| | *Collaborative Learning* | 1 | 2 | 0 | 0 | 0 | 3 |
| **Presentation** | *Details* | 0 | 0 | 0 | 1 | 0 | 1 |
| | *Accessibility Information* | 0 | 0 | 0 | 1 | 1 | 2 |
| | *Interactivity* | 1 | 0 | 1 | 0 | 0 | 2 |
| **Learning Type** | *Learning Curve* | 6 | 4 | 1 | 6 | 1 | 18 |
| | *Creativity* | 2 | 0 | 1 | 0 | 0 | 3 |
| **Content Understanding** | *Spatial Abilities* | 0 | 2 | 1 | 1 | 0 | 4 |
| | *Memory* | 1 | 0 | 0 | 2 | 0 | 3 |
| **Reduced Costs** | *Reduced Costs* | 0 | 1 | 0 | 1 | 0 | 2 |

Table 1: Benefits of Augmented Reality in Education Environments (Diegman, 2015)

## PROBLEM SOLVING WITH AUGMENTED REALITY

About the technology and problem solving: ‘the appropriate use of technology for many people has significant identity with mathematics problem solving’ (Wilson, 1993). About the augmented reality based problem solving, teachers and students reported that the technology-mediated narrative and the interactive, collaborative problem solving affordances of the AR simulation were highly engaging, especially among students who had previously presented behavioural and academic challenges for the teachers (Matt, 2008).

A new level of problem solving is achieved with the help of AR and it needs a constructive learning environment. Strommen uses the expression of ‘child-driven learning environment’. The two key elements of the process are the changed teacher-student relationship and the curriculum based using rich resources and active problem solving.

A possible definition of the concept of constructive learning environments was defined by Wilson (1995):’A place where students can work together and help each other by using diverse tools and information resources to reach learning goals and problem solving activities.’

Such a constructive learning environment (Jonassen et al., 1999) is based on the following elements: environment of problems, prior knowledge, information sources, cognitive tools and tools for collaboration.

Seymour Papert describes constructive pedagogy as constructionism and sees its implementation with the help of IT tools. Papert considers the organizational system and the methods of traditional education obsolete and extremely inefficient. When formulating his criticism, he begins with the idea that young children develop their inner world of knowledge by continually ‘scanning’ their surroundings. This direct and personal way of acquiring knowledge is generally successful as children have a passionate interest in their outer world. Papert believes that the possibilities of education in the past were very limited due to the poor range of available media and the rigid organizational and overly tied content system of education. Today, with the help of the extremely expanded media opportunities, it is possible to adapt learning to the individual ‘mental style’. The most important thing is to develop an environment in which children can learn on their own in accordance with their individual interest and cognitive style. The goal is to reach as much learning as possible with the least teaching (Papert, 1988).

Our research also took place in such a constructionist learning environment where geometric shapes hidden in callers formed the problem. The task was to define their names and attributes. We made tablets available to them with the application for decoding call pictures. Using their prior knowledge, they solved the tasks by using the Internet and their cognitive skills.

## AR APPLICATIONS AND THEIR USE FOR MATH

Nowadays, the augmented reality technology is available for the users of mobile devices. With schoolchildren it is extremely motivating, if we make fields which are difficult to understand, such as geometry, more comprehensible for pupils with the aid of augmented reality. Our research carried out with the Augmented Polyhedrons application deals with the introduction of an application into primary schools, which portrays polyhedrons in 3D.

### The list of math AR app

We present some applications that can be used both in the course of the classroom and out-of-school activities. We have tried these apps both in the classroom and at home, thus in this paper, we use our self-made photos (figure 1-7). Using the identified and described benefits in Benefits of Augmented Reality in Education Environments (Diegman, 2015) research, we present the benefits of all applications.

### *1. Quiver – Platonic Solids*

Platonic Solids is the part of the Quiver Education application.With help, we can visualize the five Platonic Solids: Tetrahedron, Hexahedron, Octahedron, Dodecahedron and Icosahedron. It is interesting that on the pages we see only the surface of the bodies and with the help of the app, we can present them in 3 dimensions followed by the appearance of the four classical elements (earth, water, air, fire) and the fifth element aether, added by Aristotle.


Figure 1: The hexahedron with different drawings on its six faces

To sum up, the test of this app was impressive, and we think that it is more than an app which presents the polyhedrons. It is not only geometry, but much more, it also teaches the history of science. The benefits of using this app are the improvement of the learning curve, creativity, spatial abilities and memory.

### *2. AR Flashcards Addition and Shapes*

This is an application family. We tried the AR Flashcards Addition and AR Flashcards Shapes applications. After printing the pages of cards, we had to purchase the two applications.

In our opinion, the Addition app and cards clearly illustrate the addition of digits from 0+1 to 9+9. It is possible to bring several cards to life at the same time.

We must also tell that some cards badly show the animals, for example on the card, a giraffe can be seen, and the app presents lions.

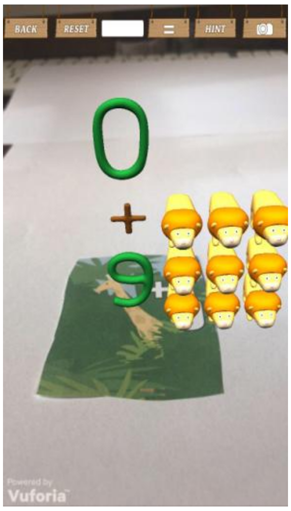


Figure 2: Appearance of lions instead of giraffes

The AR Flashcards Shapes application is very good, and we have more options to try. When we are colouring the shape, we can hear the name of the colour and the name of the shape in English.

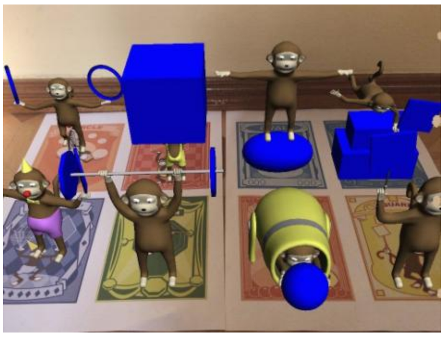

Figure 3: Multiple shapes at the same time

The benefits of these AR apps, which we emphasize, are the increase of motivation and information accessibility.

### *3. Aurasma*

We can use Aurasma to make our own AR display or video. For instance, we solve a math problem and can make a video of this process. After that we can add this video to that problem (maybe a photo of the exercise) with the help of Aurasma. It is very good for the schoolers who are interested in making their own AR world's themselves. The use of Aurasma is particularly recommended for the constructive method.

### *4. Augmented polyhedrons – Mirage*

It is extremely motivating in case of schoolchildren if we make fields, which are difficult to understand such as geometry, more comprehensible for them with the aid of augmented reality. Our research, which was carried out with the Augmented Polyhedrons application, deals with the introduction of this app into primary schools. You can read about it more detailed in the second part of the article.

Figure 4: Schoolchildren represent the geometrical shape with Augmented Polyhedron app

### *5. Stack Ar*

This is a game. The task is to stack up the blocks as high as you can. This is not a special education app but it visualizes the construction of the tower and at the same time it improves spatial abilities. We put the blocks in the real word, for example the tower appears in our room, on our table with the help of augmented reality. Using this app, we can increase attention and concentration.


Figure5: The tower on the table made by Stack AR

### *6. PhotoMath*

It is a 2D AR app. With PhotoMath, we point our camera toward a math problem and it will show the result with a detailed step-by-step instruction. What is really exciting about it is that this app contains handwriting recognition. It is absolutely useful that we can explore beautiful graphs. The app can be used for self-checking and visualisation.


Figure 6: Resolution of a handwritten equation system.

There are many other applications in this segment of education but we just focused on the well-known and used ones.

## RESEARCH – CASE STUDY

'*It keeps track of what we had to figure out by ourselves, and we can follow this track again if we need it.'* – G. C. Lichtenberg: Aphorismen

Our research was carried out in one of the fifth year elementary school classes (5.A) at Hlbokacesta street primary school in Bratislava on 16th June 2017. There were 23 10-11 year-old students in the classroom. Pupils worked in pair during the lesson using the constructionism environment.

Learning problems require time (months), collaboration, creativity and learning new competencies. Design thinking as constructionist learning is really an object of didactic interest. Constructionism enables the realization of a learning process that is designed as a developed entity for the elaboration of information into intelligent human thoughts, products and artefacts, which have intellectual (idea), manual (materialistic product / artefact) and expressive (artistic product / artefact) form. The learning and learning process is created by learning subjects (learners) together with the teaching subjects – they take the role of designers. They are the protagonists of the active elaboration of information gained from the outside and use their skills to identify and acquire relevant information, together with the ability to search for information acquired in the learning problem, learning topic/topic. Building in a learning and learning process is an activity of integrating intelligence (in order to deal with problems and be creative), in which subjects make an effort to transcribe and translate relevant information (as an important entity) into their own mental activity. It is primarily about the process of transforming the known into the unknown and vice versa, resulting in various versions of knowing / understanding. Thus students learn in an active and conscious way, which leads to more learning than is required from the subject matter itself (Kostrub, 2017).

**Research questions:**

- Does the use of the AR Polyhedron app affect the problem-solving competence of children, if yes how? In addition, we wanted to see how they struggled with other (IT, language) issues.
- Does the use of AR application motivate children to search and understand of geometric concepts?

**The methodology of the classroom research:**

Our research is a case study based on constructivism. The participants in the case study are children between 10 and 11 who solve mathematics and IT problems in groups of two or three in math lesson.

The teacher did not give any help. The children worked constructively by learning from each other and discovering the solution.

**What happened in the classroom?**

The geometrical shapes were the topic of the lesson on which their earlier knowledge was limited to the name of the various bodies, but their features were unknown for them. The task is as follows: *With the help of the AR application available, display the geometric shape hidden in callers and then name and describe its feature.*

At the beginning of the activity, the pupils select randomly a few cards with a code that the application recognizes. Using the Polyhedron application on the tablet or on the phone, they can use the augmented reality to study the geometric shape that appears in three dimensions.

During the research, the task of the pupils was to name what form they could see, and to tell as much information as possible on the basis of their observations.

The task took 20-25 minutes and it was a pair work without any support of the teacher because one of the essentials of the research was to make them discover both the use of the device and the world of geometric shapes. After the students asked if they could use the Internet, the teacher allowed to use it in the course of the problem solving. A particularly interesting part of the activity was to see how students tried to find those geometric shapes on the Internet which were unknown for them. They were looking for pictures to identify the shape. Occasionally, only the English name was found in case of truncated pyramid and truncated cones. The 25-minute work was a good example of constructionism learning when students discovered and used tools and knowledge available, such as tablet, smartphone, internet and their language skills.

They discussed and thought over several times (using critical thinking) if their thinking and their solution to the problem were certainly good. If they were not sure, they displayed again the geometric form with the help of AR application and counted the number of edges, vertices, and faces.

In the other part of the lesson, the groups presented each other the new knowledge and opinions while discussing their arguments.

As a conclusion, children were motivated by the new method, they cooperated very well and learning was constructive. They gained new knowledge, collaborated, discussed their observations, argued and reasoned. The difficulties of problem solving were well managed, they had logical ideas.

**Evaluation of research**

Our qualitative research relies on the description of the teacher's observations and on the video recording of the students' work.

Figure 7: They discover the features of the shape

Videos were made during the lesson analysed by ATLAS.TI.

In our qualitative research, matching the research questions with the analysis of videos, the codes were grouped into the following categories and subcategories:

1) Category: problemsolving competence of children
   a) Children recognize and name the shapes they already know (cube, rectangle, sphere, cone, pyramid, cylinder)
   b) Children can find the names of their geometrical shapes (triangular prism, irregular prism etc.)
   c) Children are able to count the vertices of the shape, the edges and the faces in the 3-dimensional image
   d) Children were able to search the information about shapes on the internet (naming, volume, surface formula, etc.)

2) Category: motivating children to work
   a) Children are very eager to work and motivated by the activity
   b) Children teach each other, consult, argue, counteract

Through the use of the open coding method and video-recording transcription, we defined categories and their sub-categories which are summarized in the following table 2. These expressions, told by pupils were selected from the videos.

| CATEGORY | SUB-CATEGORY | EXPRESSIONS |
|---|---|---|
| 1. Problemsolving competence of children | a. Children recognize and name the shapes they already know (cube, rectangle, sphere, cone, pyramid, cylinder) | This is a cube.<br>This is a prism.<br>Look, this is a button.<br>Look, I caught the pyramid.<br>Is this a cube or cuboid?<br>Are you sure this is a cylinder? |
| | b. Children can find the names of their geometrical shapes (triangular prism, irregular prism etc.) | Let's find out what is it called.<br>Let's look in the textbook what it is.<br>This is similar to the cuboid, but it is not, let's check it on the web.<br>What shall I enter in the search line, shapes? |
| | c. Children are able to count the vertices of the shape, the edges and the faces in the 3-dimensional image. | Let's count how many vertices it has.<br>Turn it around because I am counting its faces.<br>This pyramid has four faces. |
| | d. Children were able to search the information about shapes on the internet (naming, volume, surface formula, etc.) | What is the net of the cuboid?<br>Let's check it on the internet.<br>How to calculate the area of the prism? Can you look it up?<br>I've found it! Here is the area and volume of all the shapes. Let's find our shape. |
| 2. Motivating children to work | a. Children are very eager to work and motivated by the activity | How good it is! I touched the cone!<br>Let's see it as well. |
| | b. Children teach each other, consult, argue, counteract | We are going to …, so that we can have, we will make, we have made, we have named, we have built, we are talking about, we are colouring, we are going to write…<br>we are solving together |

Table 2 The analysis of the recorded chatting

## CONCLUSION

Finding the solution means finding the relationship between things or thoughts treated separately so far. (Among things we have and which we are looking for, between data and unknown, hypothesis and conclusion.) (Pólya, 2010).

There is a close link between problem solving and constructivism. Children's problemsolving skills can only be really developed in a creativity-promoting constructivist, or even in a constructionist environment.

The constructionist design of teaching involves constructionism, design and architecture. It is a conceptual art shared jointly and mutually by the students and their teachers as a result of the discursive and narrative essence of teaching in which they make proposals, discuss, describe, present, approve, evaluate, judge, agree, create and rework the non-material form of their products into a material form and vice versa. This takes place in the form of intentional but indirect student and teacher participation in teaching activities controlled by the teacher, in which, however, the teacher acts as a consultant when he/she is invited by the students to comment on their ideas, while avoiding any reference to mistakes. Constructionist teaching takes place through didactically considered but conceptually open teaching activities, and through discourse (controlled argumentation, handling facts) in the form of individual as well as group exploration (learning groups), thanks to which common knowledge and understanding is established.

In implementing, analyzing and evaluating our research, we took into account our objectives and research questions that we had defined at the beginning. Therefore, during the course of our research, we focused on analyzing the use of mobile technology and AR app in the teaching and the problem solving of mathematics to 10 and 11 year old students. Thanks to the consistent transcription and analysis of video-recordings and records of observation, we were able to answer our research questions.

Our first research question was: Does the use of the AR Polyhedron app affect the problem solving competence of children, if yes how? In addition, we wanted to see how they struggled with other (IT, language) issues.

This question can be positively answered on the basis of our case study. We have observed that by the usage of AR application, the problem solving ability of the children could manifest. They were also willing to solve non-mathematical problems, they tried to answer all the questions as much as possible, they learnt from each other, solved mathematical and IT issues in a cooperative manner. There was no child in the class who would have avoided any of the tasks.

Our second research question was: Does the use of AR application motivate children to search and understand of geometric concepts?

We got an even clearer answer to this question. As psychologists have already proved (Bujak, K. R. et al., 2013; Di Serio, Á. et al., 2013; ), the motivation of children in the classroom has increased greatly merely by the presence of AR application. This is not only attributed to its novelty, but also to the nature and quality of that.

In the future, more and more complex research is needed, the qualitative research should be followed by a quantitative one. By the end of our qualitative research, several hypotheses had emerged that we would like to test and prove later on.


### *Acknowledgment*

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# MATHEMATICS TEACHER TRAINEES FACING THE „WHAT-IF-NOT" STRATEGY – A CASE STUDY

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*This study is guided by the question: Can a systematic instruction in problem posing strategies increase teachers' flexibility in the way they look at textbook problems? The participants in the study consisted of group of math teacher trainees, in the last semester of their training. They got experiences with problem posing strategies during their training. After solving a textbook problem, students explored the problem more deeply by using the „What-If-Not" strategy. Three different solutions appeared in the classroom. I analyse these approaches on the basis ofPólya's steps of problem solving and Ernest's categorization of inquiry methods.*

*Keywords: Problem posing, What-If-Not strategy.*

## INTRODUCTION

In this paper, I focus the role of problem posing in mathematics teacher training. The Hungarian curriculum for teacher training [8/2013. (I. 30.) EMMI rendelet] says very little about problem solving and problem posing:

> [The teacher knows about the following subjects.] Opportunities to develop problem-solving skills, step-by-step approaches, and strategies. Classification of mathematical problems and tasks according to various aspects, problem areas, problem variations. Problem-solving and problem-building process. Types of word problems, solving steps and methods.

I report here a class for mathematics teachers' trainees in the last semester of their training. The instructor in the classroom was myself. I taught this group for five semesters including a problem-solving seminar and four courses on mathematics didactics. I included problem posing episodes regularly during their training.One form of these episodes followed to problem solving activity, and the other form was based on situations and information materials. In other words,problem posing was incorporated as posing problems related to a given problem and posing problems from a set of given material.At an early stage of problem reformulation, we applied only „surface methods", i.e. the structure of problem remains unchanged. Later we used more sophisticated problem formulation methods.The deep analysis of this long-term program is out of the scope of this paper, but generally my experience is likethe results described in(Grundmeier, 2015).In the early stage of problem posing practice students get frustrated by this activity which was new for them, but the group was very open-minded and cooperative toward novelties. As participants gained problem posing experience they became more efficient problem posers and became more creative at problem reformulation.

In the lecture reported here, firstly students solved a problem from a textbook for 7th-graders (Kovács, Szeredi, & Földvári, 1988). After that, in group work, students explored the problem more deeply by the „What-If-Not” strategy. The textbook itself is not in use today in schools in Hungary, but for pedagogical reasons I use it regularly in teachers' training, because it represents the inquiry oriented approach.Namely it was the result of the Complex Mathematics Education Experiment in the sixties-seventies in Hungary guided by the famous VargaTamás (1919-1987).

**Research questions**

My main research question here is whether a systematic instruction in problem posing strategies can increase teachers' flexibility in the way they look at textbook problems. I consider this question important because of the inflexible attitude of many teachers toward textbooks. As Brown and Walter (1983) argues „there is a myth that it is the role of the expert or authority to ask the questions and for the student merely to answer.” Moreover,Tichá and Hospesova(2013) reported that pre-service teacher education students enter university with naive ideas about the nature of mathematics and mathematics education. The authors argue that change in this view can be fostered by activities associated with problem posing, particularly when these activities are followed by joint reflection on the problems posed.

Another research question is how to cope with the open nature of problem posing episodes in classroom situation. The traditional problem posing activities are quite open. However, in the real classroom every lesson has its own goal, and possible negative effects of problem posing i.e. implausible questions, indeterminate or ponderous problem variations may take the lesson off track. Moreover, problem-generation tasks may add considerable time to the teacher's lesson plan. So, description of interactions between the students' proposalsand the teacher's agenda is another aspect of this study.

**Theoretical underpinnings**

In his famous book Pólya (1957)suggested that „The mathematical experience of the student is incomplete if he never had an opportunity to solve a problem invented by himself. The teacher may show the derivation of new problems from one just solved, and doing so, provoke the curiosity of the students”. Although the phrase „problem posing” refers to both the building of new problems and the re-formulation of given problems, Pólya clearly focuses on the „re-formulation” part of the process, and I will restrict my study to this part, too.

In his review article Silver (1994) considers many perspectives of problem posing. Here, I mention three of them, which constitute the foundation of my teaching practice.

Problem posing is a feature of inquiry oriented instruction.
Problem posing is a means to improve students' problem solving skills.

Problem posing improves students’ attitude toward mathematics.

I will review these aspects in detail.

Ernest (1991) discerns three forms of inquiry methods for teaching mathematics: guided discovery, problem solving and investigatory approach, see Table 3. Problem posing is the decisive part of all of them, but only in the third model (investigatory approach) belongs to the students’ tasks. In this case the teacher chooses the starting situation only. Inquiry-oriented approach involves increased learner autonomy and self-regulation.

| Method | Teacher’s role | Student’s role |
|---|---|---|
| Guided discovery | Poses problem with goalin mind. Guides studentstoward goal. | Follow guidance. |
| Problemsolving | Poses problem. Leavesmethod of solution open. | Finds own way to solve problem. |
| Investigatoryapproach | Chooses starting situationor approves studentchoice. | Defines own problemswithin situation. Attempts to solve inhis/her own way. |

Table 3. Inquiry methods after (Ernest, 1991) , p. 286.

The connection between problem solving and problem posing is based on the fact that problems are continually being reformulated while they are being solved (Kilpatrick, 1987). The „psychological side” of this statement, i.e., that a problem is not a problem for you until you accept it and interpret it as your own, goes beyondthe scope of this paper. Concerning the „mathematical side”, as many authors point out, Pólya’s heuristic advices give a good framework for the study of connection between problem solving and problem posing(Leung, 2013). Kilpatrick discusses the planning phase where the problem solver may modify the conditions in the problem or breaks the problem up into two or more subproblems. „The newly formulated problem may be only a stepping stone of the original problem, or it may turn out to be a problem of considerable interest in its own right.”

Pólya’s looking back phase is also a natural ground for problem creation. After a problem has been solved, the solution can suggest additional problems. The phase of looking backleads not only to generalizations but has its own role if a problem hasbeen solved incompletely or in a cumbersome way: formulating the original problem in another way might lead you to a better solution (Kilpatrick, 1987). Moreover, a new set of questions in the looking back phase illuminates the significance of what the problem solver has done (Brown & Walter, 1983).

There are various aspects of the problem posing activity in students' attitude toward mathematics. From the point of view of teachers' training I consider Silver's view who argues that problem posing frees students and teachers from the textbook as the main source of wisdom and problems in school mathematics (Silver, 1994). Moreover, the investigational approach deemphasizes the uniqueness and correctness of answers and methods andemphasizes instead humans as active makers of knowledge (Ernest, 1991). Therefore, regular problem posing practice is a means against the usual students' belief that they are passive customers of others' mathematics. This belief was also recognizedby Schoenfeld (1988).

The „What-If-Not" strategy is explained in (Brown & Walter, 1983).This approach emphasizes the generation of unfamiliar problems from previously solved problems or from theorems by varying their conditions. The essence of the strategy is that after identifying the underlying mathematical structure of problem, we modify one or more of its components. The basic implementation of the process consists of four plus one levels:

Level 0. Choose a starting point (problem or theorem).
Level 1. List some attributes of the problem.
Level 2. Ask for each attribute "What-If-Not?" that attribute.
Level 3. Pose questions.
Level 4. Analyse and answer the new questions.

**Participants, data collection**

The lessons reported in this paper took place in the Didactic of Mathematics course held on March 2, 2017 and March 9, 2017. The course was attended by eight prospective mathematics teacher students of the University of Nyíregyháza.The seminar was led by the author of this paper (hereinafter the instructor).Here I mention the students by their first name.

The activity reported here took 45+25 minutes. (The last part of the lesson on March 2, and first part of the lesson on March 9, where the homework was discussed.) Immediately after the lessons I wrote detailed reports. During the lessons, I took photos of the whiteboard. For the analysis, I also used the students' notes.

**Analysis of the problem solving activity**

**Problem No. 166** from (Kovács, Szeredi, & Földvári, 1988).

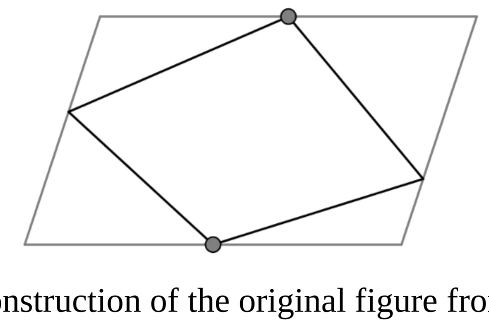

Figure 1. Reconstruction of the original figure from the textbook

Connect the midpoints of two opposing sides of the parallelogram with arbitrary points on the other two sides (Figure 1). Show that the area ofthe quadrilateral defined by the four connecting segments is half of the area of the parallelogram!

I set this problem in the last part of the topic „teaching geometry”. My generalaim to work on this problem was presenting an inquiry-orientedapproach toa problem in geometry and emphasising Pólya’s looking back phase.A more direct intention was to collect some elementary tools that are useful to solve problems on area.

Moreover, my first association as I read the problem was the Varignon’s theorem. (The midpoints of the sides of an arbitrary quadrilateral form a parallelogram. If the quadrilateral is not of crossing type then the area of the parallelogram is half the area of the quadrilateral (Coxeter & Greitzer, 1967). I was not sure whether students have known this theorem or not, and I wanted to insert this statement in the class discussion.

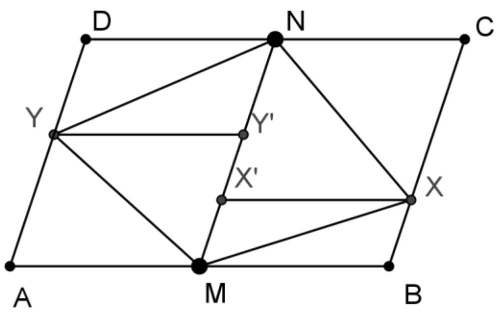


Figure 2. First solution: the „dissection strategy”

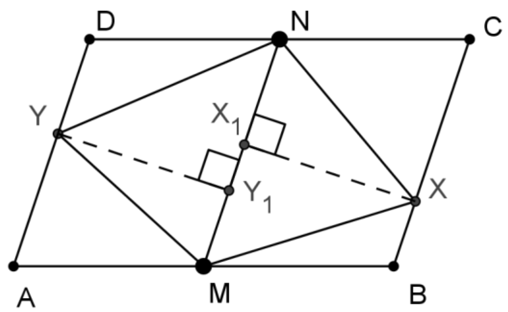


Figure 3. Second solution: the „altitude strategy”

This exercise was not a real problem for my students, and two different solutions appeared during individual activity immediately.These strategies appear later in the class; thus, I label them with „dissection strategy” (Figure 2) and „altitude strategy” (Figure 3). In Figure 2$XX' \parallel AB$ and $YY' \parallel AB$, andtriangles $AMY$, $MBY$, $XCN$ and $NDY$ can be rearranged to fill the quadrilateral $MXNY$.The other solution (Figure 3)was that the altitude of the triangle $MNY$from$Y$ equals to the altitude of the parallelogram $MNDA$ perpendicular to $MN$. (Analogously for $MXN$ and $MBCN$.) Note that the teachers’ edition uses the „altitude strategy”.

From the point of view ofinquiry based learning, this part was a problem solving activity in the classroom. The instructor’s role was the problem posing, and he leaved the method of solution open.The students found their own way to solve the problem. Otherwise, this activity was at the „Level 0” phase in the process of problem posing. The instructor specified the starting point of further discussion.

## Analysis of the problem posing activity

### *Level 1 of problem posing*

At this stage the instructor presented the aim of next activity: posing new problems based on the original problem. In a classroom discussion, the students collected the attributes of the problem. The following features appeared

1. A quadrilateral is given.
2. A parallelogram is given.
3. Two midpoints appear.
4. Two arbitrary chosen points appear.
5. The statement is about the area.

The first four attributes refer to the given formation, but attribute 5 refers to the statement of the problem.

### *Level 2 and 3 of problem posing*

In group work students posed „What-If-Not" questions. Three different approaches appeared:

**Group 1**: Connect *trisecting points* of two opposing sides of the parallelogram with arbitrary point on the other two sides.In their notebook, the trisecting points were chosen as shown in Figure 4.

They kept attribute 2 of the problem, and they altered attribute 3.

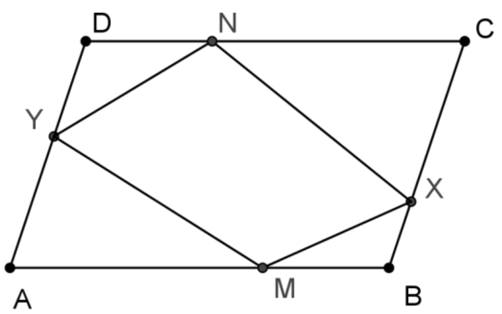


Figure 4. $MB = ND = {}^1/_3 AB$

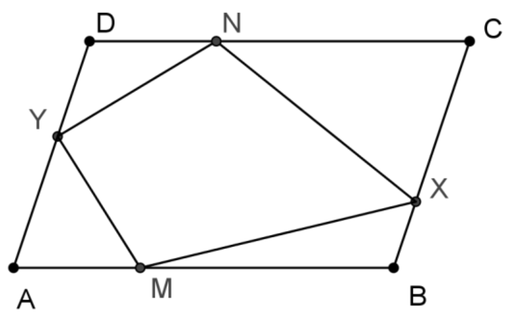


Figure 5. $AM = ND = {}^1/_3 AB$

**Group 2**: Connect the midpoints of two parallel sides of symmetric trapezoid with arbitrary points on the other two sides.

They altered attribute 2, and they kept attributes 3 and 4.

**Group 3**: Chose the midpoints of two opposing sides of a regular hexagon and arbitrary points on other sides. Look for the convex hull of the six points.

They altered attribute 1, and they kept attribute 3 and 4.

Students' activity corresponds to Level 2 of problem posing. A common characteristic of their work was that it did not contain statements or questions. At this point students fixed the new formation, but they were unable to make observations or conjecture. In classroom discussion, I encouraged them to put questions and they proposed a rather obvious question: *Is the original statement true or false*?

### *Level 4 of problem posing: the case of trisection points (part 1)*

The instructor asked whetherFigure 4 represented the only way to chose the trisection points or not. They immediately found the other possibility, which is shown inFigure 5.Balázs noticed that the dissection strategy works even in this setting. In addition, the only essential condition is that $MN \parallel AD$, or equivalently ${}^{AM}/_{MB} = {}^{DN}/_{NC}$. We clarified that the altitude strategy works as well. At this point we arrived at the first problem variation.

**Problem 1.** *Let $ABCD$ a parallelogram, $M$ and $N$ are points on parallel sides $AB$ and $CD$ such that $MN \parallel AD$. Connect $M$ and $N$ with arbitrary points on the other two sides (Figure 5). Show that the area of the quadrilateral defined by the four connecting segments is half of the area of the parallelogram!*

### *Level 4 of problem posing: the case ofa symmetrical trapezoid*

This variation needed a more direct intervention from the instructor.

**Instructor:** The area of the inscribed quadrilateral depends on two points $X$ and $Y$. How to get experience of this function?
**Orsolya:** Put $X$ and $Y$ in extreme positions!
**Richárd:** Put $X$ and $Y$ in a special position, e.g. chose the midpoints of the legs!
**Instructor:**Let $m$ denote the altitude of the trapezoid, $AB = a$, $CD = b$ and $a > b$! Make and analyze the corresponding figures!

Students created Figures 6 to 9 indicating „extreme positions". They compared the area $t$ of the inscribed polygon with area $T$ of the trapezoid, and they found that the original statement is no longer true. I encouraged them to put a conjecture for the area of the inscribed polygon.They concluded that extreme positions for the location of points $X$ and $Y$ perhaps indicate extremes for the area, as well. In individual work the following proof appeared. Segments $XX'$, $YY'$from the „dissection strategy" (i.e. $XX' \parallel YY' \parallel AB$, see Figure 10) are altitudes of triangles $MXN$ and $MNY$ corresponding to the side $MN$. Moreover, $\frac{a}{2} \geq XX' \geq \frac{b}{2}$ , $\frac{a}{2} \geq YY' \geq \frac{b}{2}$ and these inequalities give the statement: $\frac{1}{2}m \cdot b \leq \frac{1}{2}m \cdot XX' + \frac{1}{2}m \cdot YY' \leq \frac{1}{2}m \cdot a$, i.e. the area of the inscribed polygon is between $\frac{1}{2}m \cdot b$ and $\frac{1}{2}m \cdot a$. This thread led to the second problem variation.

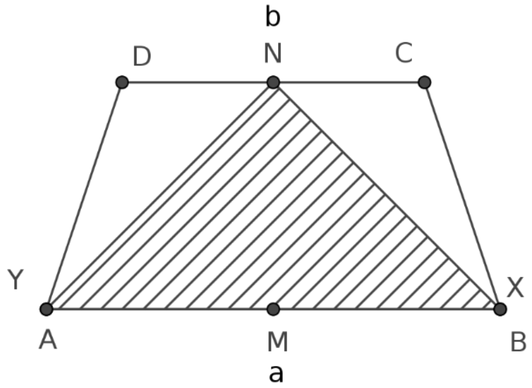


Figure 6. $\boldsymbol{2t = a \cdot m > \frac{(a+b)\cdot m}{2} = T}$

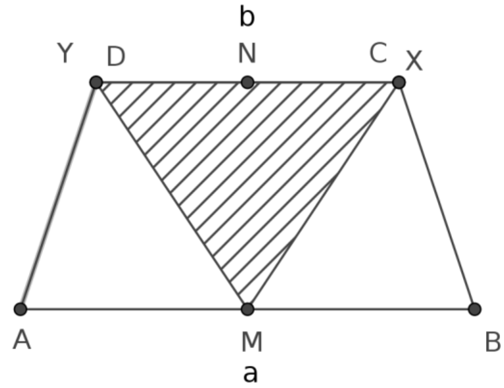


Figure 7. $\boldsymbol{2t = b \cdot m < \frac{(a+b)\cdot m}{2} = T}$

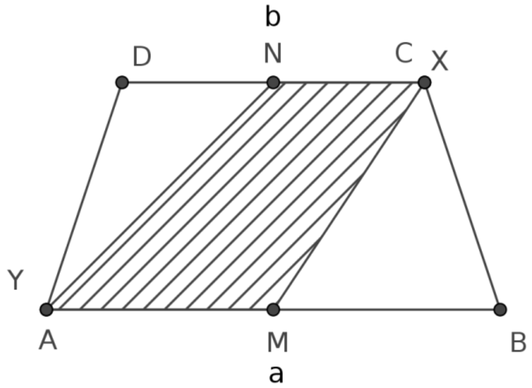


Figure 8. $\boldsymbol{2t = \frac{(a+b)\cdot m}{2} = T}$

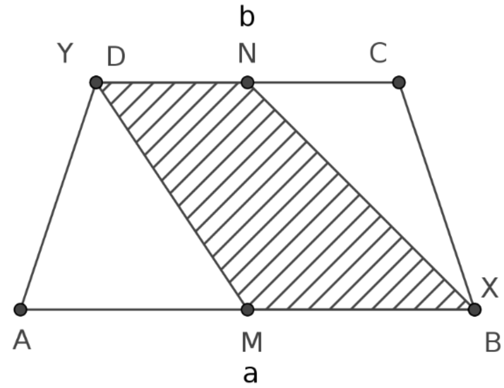


Figure 9. $\boldsymbol{2t = \frac{(a+b)\cdot m}{2} = T}$

**Problem 2.** *Connect the midpoints of two parallel sides of a symmetrical trapezoid with arbitrary points on the other two sides. Prove that the area of the inscribed polygon is between* $\frac{1}{2}m \cdot b$ *and* $\frac{1}{2}m \cdot a$, *where* $a$ *and* $b$ *denote the bases of the trapezoid, and* $m$ *denotes the altitude.*

After that I pointed out that this statement is true for all trapezoids.

**Problem 2A.** *Connect the midpoints of two parallel sides of the trapezoid with arbitrary points on the other two sides. Prove that the area of the inscribed polygon is between* $\frac{1}{2}m \cdot b$ *and* $\frac{1}{2}m \cdot a$, *where* $a$ *and* $b$ *denote the bases of the trapezoid, and* $m$ *denotes its altitude. Especially, the area of the inscribed polygon is half of the circumscribed trapezoid, if a = b.*

Namely, $\frac{b}{2} \leq XX' \leq \frac{a}{2}$, and $t_{MXN} = \frac{1}{2}XX' \cdot m_1 + \frac{1}{2}XX' \cdot m_2 = \frac{1}{2}XX' \cdot m$ implies that $\frac{1}{4}m \cdot b \leq t_{MXN} \leq \frac{1}{4}m \cdot a$ and analogously for the triangle $MNY$ (Figure 11).

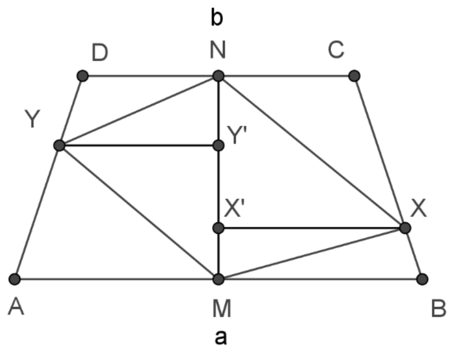


Figure 10. The dissection strategy for symmetrical trapezoid

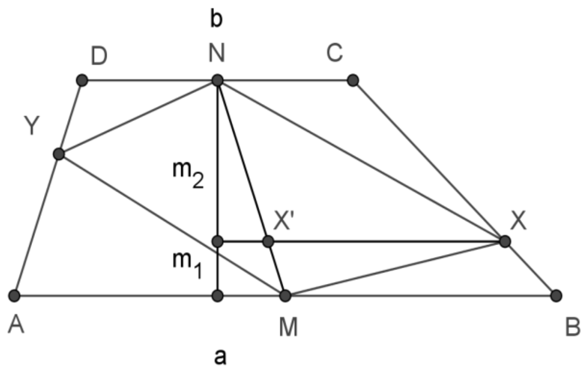


Figure 11. Problem 2A

Concerning the „special case" when $X$ and $Y$ are midpoints of the legs, they computed the area of the inscribed quadrilateral (**Hiba! A hivatkozási forrás nem található.** presents a fragment from Orsolya's notebook) without recognizing that in this case it is a rhombus. At this point we lookedout for the Varignon's theorem.

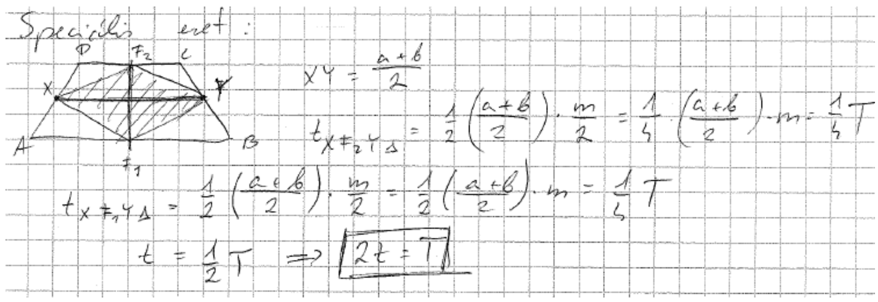

Figure 12. English translation of the text in the top left corner: „Special case"

### *Level 4 of problem posing: the case of trisection points (part 2)*

Finally, I came back to the original settings of the first group (Figure 4), and I gave it for housework. My guiding instruction was to find extreme positions for $X$ and $Y$also for this problem. Following the thread for the trapezoid, we got Problem Variation 3.

**Problem 3.** *Connect trisecting points of two opposing sides of the parallelogram with arbitrary points on the other two sides as shown in Figure 4. For area $T$ of the parallelogram and area $t$ of the inscribed polygon we have* $\frac{1}{3}T \leq t \leq \frac{2}{3}T$.

### *Level 4 of problem posing: the case ofa regular hexagon*

There was no time to complete this task in the classroom, so I gave the problemas homework with the guiding instruction that only a special case should be addressed: all the points of the inscribed polygon are midpoints of sides of the original polygon.In this case the inscribed hexagon is a regular hexagon also,and it is easy to see that the ratio of areas is $3/4$. The following week we discussed that this ratio remains unchanged if every second point is arbitrarily chosen on the sides (Figure 13).

**Problem 4.** *Connect the midpoints of every second sides of a regular hexagon with arbitrary points on the other three sides as shown in Figure 13. The area* $t$ *of the inscribed hexagon equals* $3/4 \cdot T$, *where* $T$ *is the area of the circumscribed hexagon.*

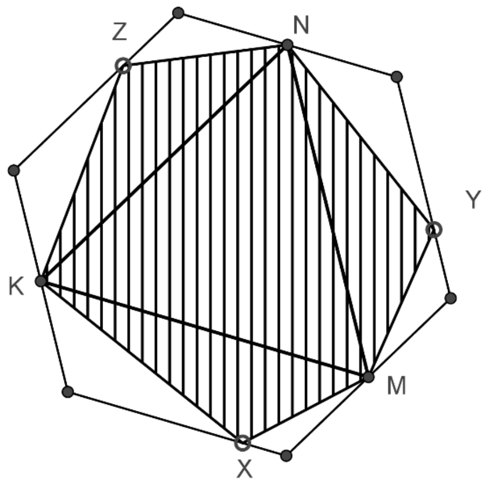


Figure 13. $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ are arbitrarily chosen points on the sides.

## Discussion

My discussion is guided by two viewpoints: the instructor's behaviour in the classroom and adescription of problem variations.

Every problem set in the classroom should serve direct aims determined by the teacher. Here, one of the direct aim was to emphasize Pólya's looking back phase in the problem solving process. The inquiry oriented instruction and especially the rather open nature of problem posing episodes may produce activities with different orientations in the classroom. Some activities may correspond to the instructor's aim, other activities may produce totally different results. Some other activities may open new perspectives that were unknown even to the instructor.We look for different behaviour patterns for the teacher to cope with difficulties caused by openness.

### *The case of Group 1*

In this issue, the instructor *slightly modified* the students' proposal, and this action led to the straightforward generalization of the problem (**Hiba! A hivatkozási forrás nem található.**.). This part of the lecture can be described as a mixture of investigatory approach and guided discovery. It was a guided discovery, because the instructor had a direct goal in mind; and the instructor's first intervention interrupted the students' own thread. However, the starting point of the instructor's intervention was the students' version of the problem. Later students revised their original version.

This case is a clear example of Pólya's looking back phase. Pólya(1957)wrote that „At the end of our work, when we have obtained the solution, our conception of the problem will be fuller and more adequate than it was at the outset. Desiring to proceed from our initial conception of the problem to more adequate, better adapted conception, we try various standpoints and we view the problem from different sides." The main attribute of the problem was not changed in this problem variation, even the method of solution was the same; we have certainly a fresh look at the original problem.

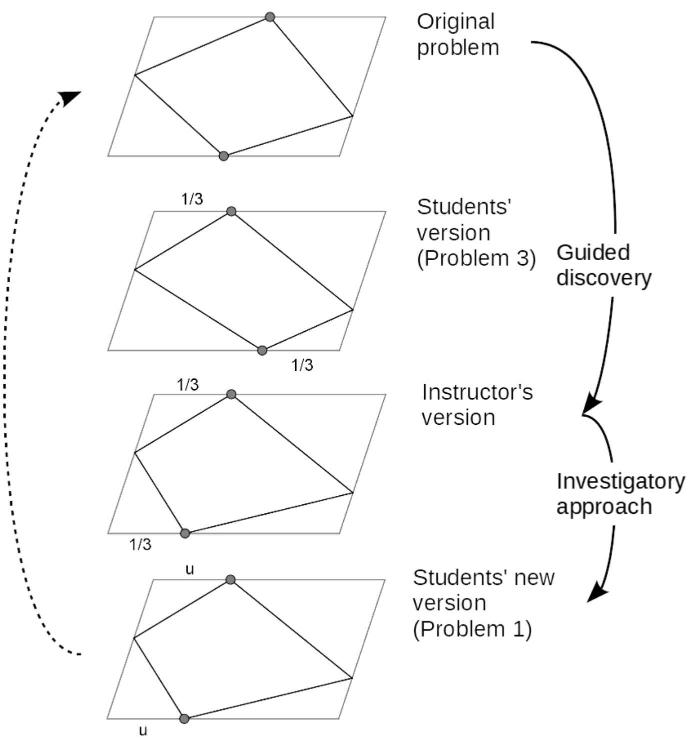


Figure 14. The activity of Group 1

### *The case of Group 2*

All the attributes of an investigatory approach (Ernest, 1991) can be identified in the first part of this progression (**Hiba! A hivatkozási forrás nem található.**). Students defined a problem within the situation, and they solved it in their own way. As the instructor recognized the prospect of further generalization, he *completed* their proposal, and at the same time he changed his behaviour to a traditional style, andhe expounded his own version. Nevertheless, this unintended and unplanned action led to a new perspective of the original problem, namely to a new proof, and completed the looking back phase. $t_{MXN} = \frac{1}{2} \cdot XX' \cdot m_1 + \frac{1}{2} \cdot XX' \cdot m_2 = \frac{1}{2} \cdot XX' \cdot (m_1 + m_2) = \frac{1}{2} \cdot \frac{a}{2} \cdot m = \frac{1}{4} \cdot T,$ and analogously, $t_{MNY} = \frac{1}{4} \cdot T$, see Figure 16.

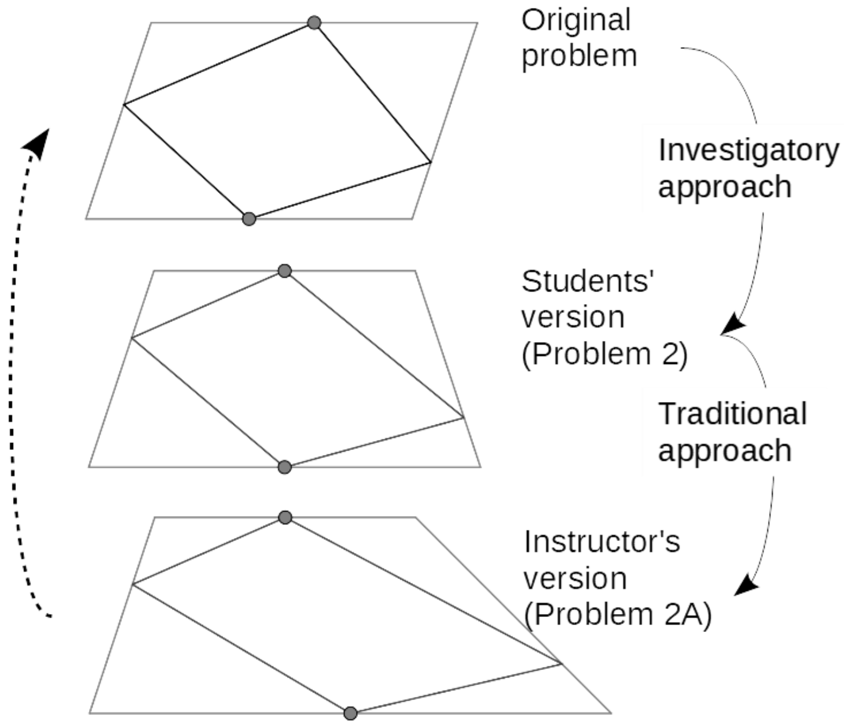


Figure 15. The activity of Group 2

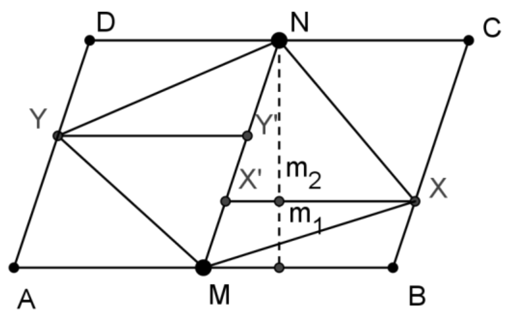


Figure 16. A new proof for the problem

### *The case of Group 3*

The previous problem variations were some kinds of generalizations of the original problem. The proposal by Group 3 can be described more precisely as a germ of analogy. However, students did not find ananalogous problem. Here, the instructor skipped the students' problem variation and *essentially modified* it. The instructor's proposal (Problem 4) was analogous in the sense that the „altitude strategy" can be used for the solution.

The orientation of the relationship between the original problem and the final problem changed. Problem 4 does not give any novelty to the original problem. Here we used the original method to a new problem which activity reflects to Pólya's

(1957) famous sentence „Can you use the result, or the method, for some other problem?”.

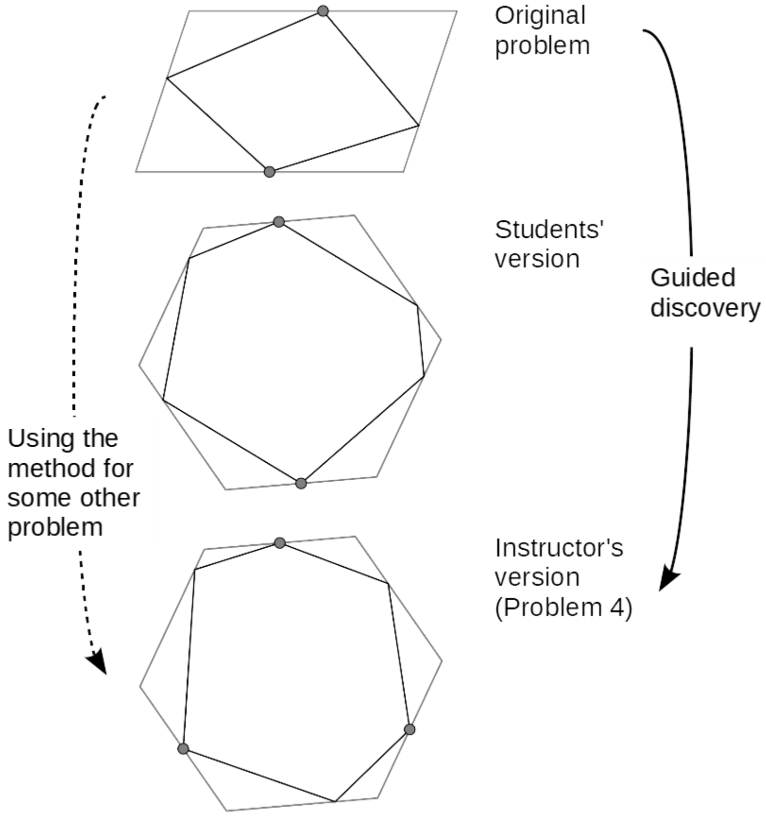


Figure 17. The activity of Group 3

In all three proposalsthere wereinterventions on the part of instructor. Thinking in terms of modes of intervention applied by the instructor, we found three types (Table 4).

| **Case** | **Relation between students’ activity and instructor’s intention** | **Type of intervention** | **Result** |
|---|---|---|---|
| Group 1 (trisection) | coherent, planned | slight modification | generalization |
| Group 2 (trapezoid) | coherent, unplanned | completion | new proof for the original problem |
| Group 3 (hexagon) | Withoutany coher-ence, unplanned | essential modification | analogous problem |

Table 4. Modes of intervention applied by the instructor

## Conclusions

This case study extends previousstudies by showing that problem-posing ability and creativity can be developedwith prospective teachers (Grundmeier, 2015). Engaging prospective teachers in problem posing results in clear consequence that they overcome the initial frustration effect caused by such type of activities and master methods of reformulating starting problems. However, formulating questions and conjectures is connected with general problem solving abilities and this fact limits the success of the problem posing activity.

Other finding of this case study concerns the inquiry style teaching.Several types of interactions between students' proposals and instructor's agenda was described here, and several types of teaching styles appeared in the seminars reported here. It demonstrated that problem posing activity can be implemented even into strictly controlled syllabus. Teachers should develop the competence to cope the unknown situation and controlling the learning process to avoid blind paths in the classroom.


## Acknowledgment

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# THE ROLES OF UNDERSTANDING AND PROBLEM SOLVING IN MEANINGFUL LEARNING

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*The fundamental idea in learning mathematics is the complementary nature of reasoning and understanding, which together develop the skills of thinking. Thinking is based on understanding, but understanding is a product of thinking. In other words, reasoning either increases understanding through new relationships or strengthens existing relationships. Therefore, it is not useful to try to separate understanding and reasoning in teaching. An important question is how reasoning and understanding are connected in the learning process. This paper is a theoretical study of the roles of understanding, reasoning and problem solving in meaningful learning.*

The development of mathematical understanding has posed vexing problems for mathematics educators in recent decades. The first problem has been to decide what ‘understanding’ means. The second problem is to distinguish it from other goals in the curriculum. The third is to figure out how to develop and promote mathematics learning through understanding.

## THEORETICAL FRAMEWORK

One of the primary tasks of the Finnish comprehensive school curriculum in mathematics is to develop pupils’ mathematical thinking and understanding (NBE 2014). Other central goals in the curriculum are reasoning and problem solving.

### Understanding

Mathematical understanding can be characterized as a continuous process that is connected with a certain person, a mathematical content domain and a special environment (Hiebert & Carpenter 1992). Understanding is defined as structured knowledge in the permanent memory, and one’s level of understanding depends on the number and strength of connections between informational elements. Mathematical understanding answers the question „Why?” and includes and activates the skills required to analyse mathematical statements. Pirie & Kieren (1994) have developed a theory of mathematical understanding as a hierarchical process of how an individual’s mathematical understanding develops. It seems clear that understanding has a dualistic nature as both knowledge and concept. If we go deeper into the concept of understanding, it becomes more complicated with many meanings (Leinonen 2011).

### Reasoning

Reasoning refers to the capacity to think logically about relationships between and among concepts and situations. It includes knowledge of how to justify conclusions. Warrant is an argument that gives logical reason as to *why* a claim is undoubtedly true (cf. Viholainen 2008, 18-21), and it is the glue that forms coherent knowledge systems. Warrant is a normative condition of knowledge because it is unworthy to believe claims without reason (Alston 1989). Adaptive reasoning refers to justification and has a broader extension than warrant. It also includes induction and intuition that could be based on informal knowledge (Kilpatrick & al. 2001).

A justified argument provides sufficient reason to believe that a claim ought to be true. The function of reasoning in learning is to add to the number and strength of connections in mental networks, and it develops conceptual understanding. According to Harel (2009), humans' reasoning involves numerous mental acts such as inferring, interpreting, explaining and problem solving.

## LEARNING WITH UNDERSTANDING

At least since Locke (1693/1995), understanding has been considered to be the power of thinking. From a pedagogical point of view, Brownell (1935) was one of the pioneers who paid attention to understanding in learning and teaching mathematics. Authors such as Ausubel (1968), Skemp (1976), Hiebert (1986) and Kilpatrick (2009) are worthy of mention as Brownell's followers. According to Ausubel NCTM (2000, 20), emphasising students' activities in learning in which understanding has to be a resource: „Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge."

Learning with understanding (LWU) and meaningful learning (Ausubel 1968) are almost synonymous and have, at the very least, an intersectional area. Koskinen (2009) has studied how the LWU tradition has changed during its hundred-year history, and he found four chronological categories:

LWOU — Learning with no understanding, or separated.

LWUA — Learning with understanding, but as an achievement.

LWUIP — Learning with understanding, as an individual mental process.

LWUSP — Learning with understanding, as a social process.

This history shows that the view of the connection between learning and understanding has changed from a discrete processes to social one. At first, understanding has minimal or no function in the learning process. Sometimes understanding is a product of learning or is interwoven into the learning process. Harel (2009) studied learning mathematics as a dualistic process in which reasoning and understanding is a complementary pair in the learning process. In Skemp's (1976) dialectic model a concept is an object of study and a problem-solving tool.

In addition to the previous notion, we can see understanding both as a framework and as a number of mental acts (Sierpinska 1994; Leinonen 2011).

## Understanding and thinking in curricula

The 1980s marked the beginning of the emphasis on teaching higher levels of thinking skills. Many publications in those times cited evidence of students' inability to answer higher-level questions and apply their knowledge (cf. Marzano 2001). One of the main consequences of this movement was to emphasise problem solving in the goals of curricula. For example, in the curriculum of Singapore (cf. Kilpatrick 2009) or the US (NCTM 2000), problem solving was the main element in mathematics. The primary goals in all subjects of the Finnish comprehensive school curriculum were to develop pupils' thinking skills, understanding and problem solving (NBE 2004).

The new Finnish comprehensive school curriculum in mathematics (NBE 2014, 428) states that the task for instructing mathematics is *to develop pupils' logical, precise and creative mathematical thinking, which creates a basis for understanding mathematical concepts and constructs and develops pupils' ability to handle information and solve problems.* That is, the primary task is to develop thinking skills, the ability to handle information and to solve problems. After the introduction in the NBE publication, more detailed goals are presented, and understanding is listed as the main goal.

NCTM (2000, 20) expresses how learning with understanding works: *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge*. The difference between the Finnish and US curricula is that the latter expresses the functional principle of learning with its essential elements.

According to the studies of Mously (2003), understanding mathematics is an important goal for teachers, but their conception of its content is poor. Therefore, the introduction of curricula could be written more specifically as follows: *The task for instructing mathematics is to develop pupils' mathematical thinking based on understanding. It creates new concepts and constructs, and it develops pupils' resources to handle information and to solve problems.*

Clarifying the key concepts such as mathematical thinking, understanding and problem solving could complement the goal-setting of curricula. It would also be useful to present a brief description of the roles of understanding in reasoning and problem solving, of which there is a more specific description in the next chapter.

## Learning with reasoning and understanding

According to philosophical epistemology, reasoning has been an essential part of knowing since at least Plato because knowledge is defined as a well-reasoned, true belief. From the cognitive point of view, reasoning refers to thinking logically about relationships among concepts and situations. Such a conclusion is reasonable, correct and valid. A correct formal proof demonstrates that a particular claim is true and shows *why* it is true and therefore warrant. This requires that a person has individual access to a subject matter, and provides readiness to evaluate a claim. But a demonstration presented in a formal form is usually not very explanatory and thus does not underpin understanding. Therefore, justification with a broader extent of reasoning has an essential role facilitating the conceptual and holistic understanding of relationships between concepts. (Kilpatrick & al. 2001; Viholainen 2008).

From an individual perspective, understanding means having an internalised knowledge or a clear and correct idea of something (e.g. Brownell 1956; Bloom 1956; Kilpatrick 2009). To understand something does not refer to an isolated object but it concerns a position in an internal network (Hiebert and Carpenter 1992, 67). The function of reasoning is to make new links between knots and to connect new information to the previous net. Learning with reasoning and understanding refers to the productive activities to develop conceptual knowledge. From a pedagogical point of view, we need a much broader conception of reasoning that includes not only the deductive formal proving and warrant but also numerous mental acts allowing informal basis.

In teaching, we have to go deeper into the concept of understanding to obtain a clearer grasp of it. For example, Leinonen (2011) introduces four modes of understanding: conceptual knowledge, grasping meaning, comprehension and accommodation. The function of those modes is to give background and conceptual tools for reasoning, to interpret the incoming information, to synthesise the knowledge, to integrate the message into previous knowledge, and to reorganise the cognitive structure. These ideas have been discussed in detail in the publication (Leinonen 2011) and in the next chapter.

**Problem solving through understanding**

Problem solving is a special process because it is present in numerous other human actions such as in abstracting, deciding, interpreting, searching or proving (Harel 2009).

Taking into account the many modes of understanding, the learning goes ahead as an interaction between knowledge systems and mental acts (Figure).

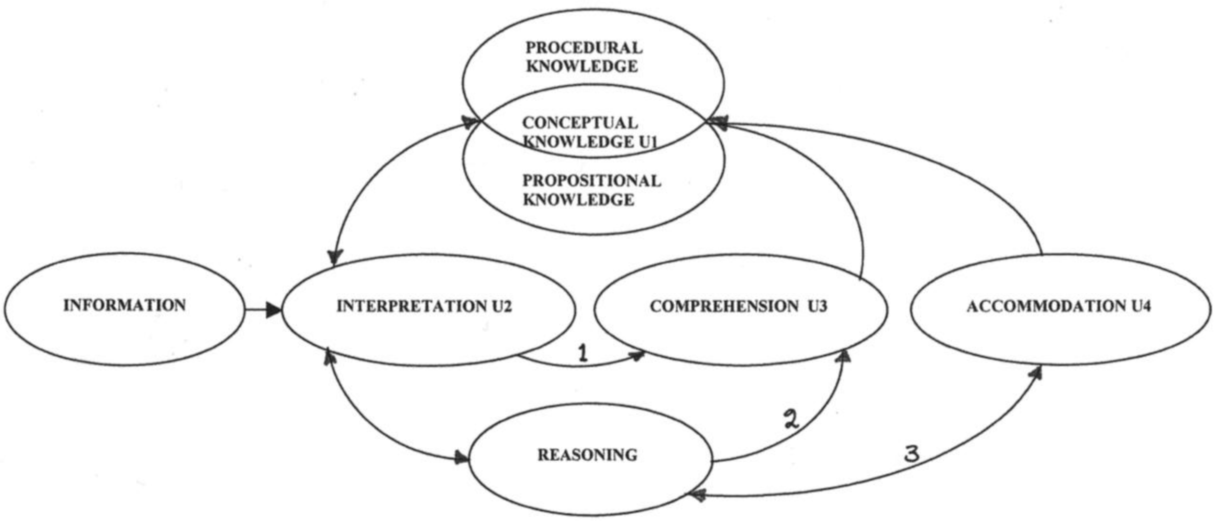


Figure. The roles of understanding in meaningful learning.

Incoming information, experiences and previously-retained knowledge are primary elements in mental actions. Meaningful learning consists of three main cycles that elaborate incoming information:

1. The associative cycle (U1 – U2 – U3 – U1): The function of this reproductive process is responsible for translating information into an appropriate form for storage in the permanent memory. The act of interpretation, U2, grasps the literal content from incoming information based on the framework of the previous knowledge, U1. The comprehension process, U3, involves two elements: synthesis and representation. The function of synthesis is to distil the incoming information to its key characteristics, organised in a parsimonious, generalised form (Kintsch 1974). The process of representation produces a symbolic analogy of the knowledge contained in the distilled structure. The concept of representation is based on Paivio's (1971) dual-coding theory combining linguistic, symbolic and imaginary modes.

2. The reasoning cycle (U1 – U2 – R – U1): The reasoning process is a cluster concept that includes various mental acts. Contrary to the previous process, this cycle requires productive activities because a learner discovers something new. This process goes beyond the literal level of incoming information. It involves identification, inferences and predictions based on information in the task and principles and generalisations already possessed by the learner. Metaphorically, the function of reasoning is to answer questions such as „How?" or „Why?" and entails, among other factors, the skills required to analyse the situation. The reasoning cycle begins to interpret U2, the task, by trying to find a relevant conception of the situation. The students confront the situation, often in the form of a question, and ponder its relation to what they already know (Burton 1992). It involves steps and heuristics associated with the situation as follows:

Identification of obstacle to goal

Identification of alternatives

Evaluation of alternatives

Determination and selection of alternatives having the highest probability of success (e.g. Sternberg 1987).

The next phases are to make a plan to solve the problem and carry it out. The process continues by analysing the solution and method. The final phase is to condense U3 and to integrate the product, in a relevant form, into permanent memory.

3. Accommodation cycle (U1 – U2 – R – U4 – U1): Unlike the associative cycle, which grasps the content and intention of the learning material, the process of understanding U4 reorganises the cognitive structure itself. This mode of understanding can produce new insights and construct new structures too. Piaget calls this change "accommodation" (cf. Linn & Barbules 1993), which also includes reasoning extension and has discovery processes too.

The accommodation may be a revolution when a learner's view of mathematics or even the basis of her/his worldview changes (Kuhn 1963). A learner's view of mathematics consists of a number of features that have a fundamental influence on accommodation and on other actions: beliefs, attitudes, affects, values, etc. For instance, relevant beliefs could promote learning, but irrelevant ones and contradictions cause difficulties in learning mathematics (e.g. Pehkonen & Törner 1996).

Although attitudes, values, interests and affects as well as intuition are important factors in meaningful learning and reasoning (Viitala 2015; Raami 2015), they are not included within the scope of this article.

## SUMMARY

It has been a common conception since Locke (1693/1995) that understanding is a power of thinking and a synonym for intelligence. Understanding has been one of the primary goals in mathematics curricula for almost one hundred years since Brownell (1935). Brownell & Sims (1946/1972) didn't want to define understanding exactly but they described it in many ways. One of them was: „An ability to act, feel, or think intelligently with respect to a situation." This is a vague concept of understanding from a pedagogical point of view. According to Kilpatrick (2009), the word 'understanding' means having a clear and complete idea of something. He continues that touch and sight are the two main metaphors in English for 'understand'.

Many researchers such as Hiebert & Carpenter (1992) have defined understanding as a knowledge system in permanent memory that has many advantages in human actions. For instance, understanding gives tools and frameworks to problem solving and to other activities in life. If we go deeper into the concept of understanding, it seems to be more complicated (Leinonen 2011). Understanding in meaningful learning (Figure) is a resource of thinking and a cluster of mental acts like interpretation, grasping meaning and accommodation. Reasoning-modul is the main „work shop" in this model.

Understanding and reasoning is a complementary pair in meaningful learning. *Thinking is based on understanding and understanding is a product of reasoning* (Harel 2009). Connections in the internal knowledge system are produced by justification that explains *why* a fact is the consequence of others. Understanding and thinking should be developed in balance to promote mathematics proficiency (Kilpatrick 2009). There is much to do in teacher education, because the teachers have poor knowledge of the concept of understanding (Mously 2003). Therefore, more collaboration between teachers and researchers is needed at both the theoretical and practical level.

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# PÓLYA'S PHASE LOOKING BACK IN THE CLASSROOM

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*One possible starting point for improving the problem-solving ability of students is Pólya's phase "looking back". Many experts consider it to be the most important phase in the problem-solving process.However, there seem to be only few empirical studies that explore how teachers make their students engage in looking back activities in the classroom. The data of an empirical study conducted in German secondary schools in 2016 indicate that these reflections are rather neglected in theclassroom. This article will mainly deal with the theoretical background and give a brief overview of the study.*

## Theoretical Basis

Pólya (1945) suggests that there are four phases through which all students must progress inorder to successfully solve a problem:(1) „Understanding a problem", (2) „Devising a plan", (3) „Carrying out the plan" and (4) „Looking back" (see Figure 1).

According to Pólya, during the phase looking back, the problem solver should examine the solution by such activities as checking the result, checking the argument, deriving the result differently, using the result, or the method, for some other problem, reinterpreting the problem, interpreting the result, or stating a new problem to solve (see Figure 2).

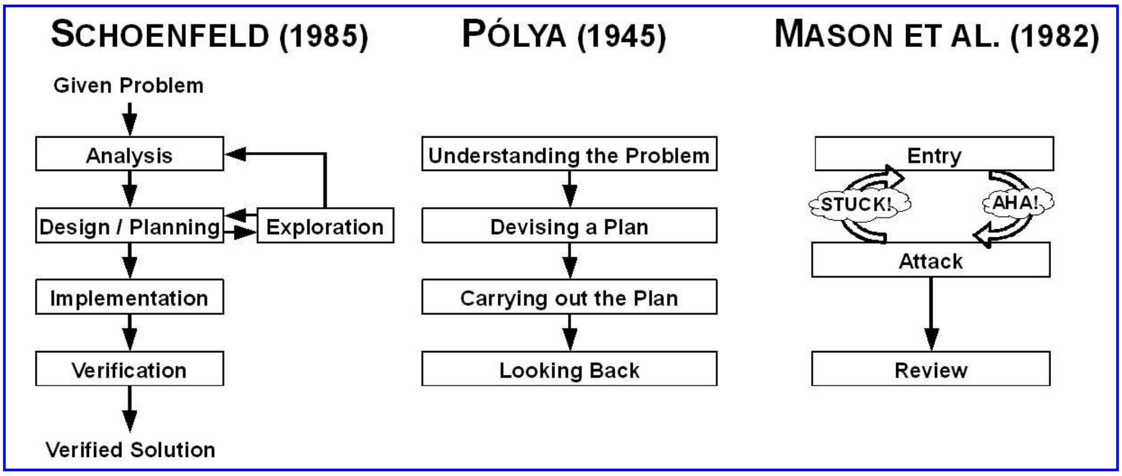


Figure 1: Models of the problem-solving process (Rott, 2013)

Pólya's model inspired many other experts who also included a final phase in their models of a complete problem-solving process that is concerned withreflections, similar to Pólya'sphase looking back (e.g. Mason, Burton & Stacey, 1982; Schoenfeld, 1985).

Mason et al. (1982), for example, subdivide the final phase called „review" in their model of a complete problem-solving process into the three subphases *check, reflect* and *extend* (see Figure 2).

| Pólya (1945) | Mason et al. (1982) |
|---|---|
| • Can you check the result?<br>• Can you check the argument?<br>• Can you derive the solution differently? Can you see it at a glance?<br>• Can you use the result, or the method, for some other problem? | check<br>• calculations<br>• arguments to ensure that the computations are appropriate<br>• consequences of conclusions to see if they are reasonable<br>• that the solution fits the question<br>reflect<br>• on keyideasandmoments<br>• on implications of conjectures and arguments<br>• on your resolution: can it be made clearer?<br>extend<br>• The result to a wider context by generalizing<br>• By seeking a new path to the resolution<br>• By altering some of the constraints |

Figure 2: Models of the phase looking back

Yimer and Ellerton (2006) developed a five-phase model that incorporates cognitive and metacognitive aspects of mathematical problem solving. This model distinguishes between the two final phases *evaluation* and *internalisation*, where *evaluation* corresponds to Masons et al. (1982) *check* subphase and *internalisation* essentially corresponds to the subphases *reflect* and *extend* suggested by Mason et al. (1982).

Wilson, Fernandez andHadaway(1993) emphasize the cyclic nature of problem-solving processes and include the individual's managerial processes such as self-monitoring, self-regulating and self-assessmentin their model (see Figure 3). The cyclic structure of this model implies that the phase looking back does not have to take place at the end of the problem-solving process but can also be connected to the other phases *understanding the problem, devising a plan*and *carrying out the plan*. Furthermore, the model shows that the phase looking back can function as a link between problem solving and problem posing. Pólya's phases are not intended to be part of a linear, step-by-step process either.Instead, they are part of a dynamic process during which problem solvers cycle back and forth between the phases (Lesh and Zawojewski, 2007; Polya, 1945).

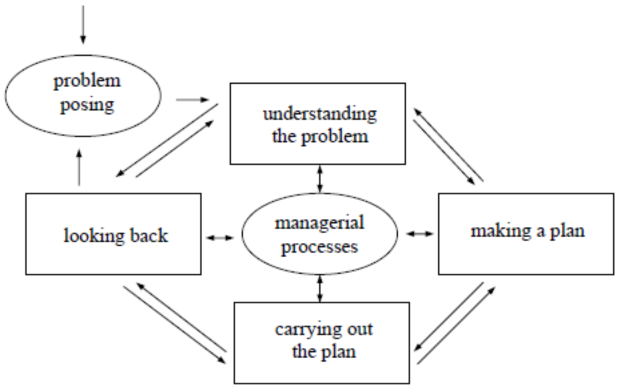


Figure 3: Cyclic model of a problem solving process (Wilson, Fernandez & Hadaway, 1993)

## The importance of the phase looking back in mathematics education

Many experts in mathematics education point out the importance of the phase looking back within the problem-solving process. Pólya (1945) and Schoenfeld (1987) call the reflection of the process a fundamental part of developing the students' problem-solving abilities.

Wilson, Fernandez and Hadaway (1993) refer to the phase looking back as the most important phase of problem solving because it provides the primary opportunity for students to learn from the problem. Kilpatrick (personal communication, October 23, 2016) states: „I believe that the phase of looking back is one of the most important – if not the most important – phases in problem solving in mathematics and therefore in teaching and learning problem solving." Kluwe (1981) and Dörner (1982) point out that problem solving depends on the constant reflection of the individual's problem solving process. Leong, Tay, Toh, Quek and Dindyal (2011) stress that the phase looking back can be seen as a link between problem solving and problem posing.

Jacobbe (2008) observes that the phase looking back is often neglected by learners of problem solving. With regard to problem solving in the classroom, Schoenfeld (personal communication, March 23, 2015) assumes that „for most teachers, 'looking back' tends to mean 'checking to make sure that my solution is correct." Hensberry and Jacobbe (2012) argue that in traditional mathematics classrooms, students are not typically encouraged to reflect on the applied problem solving strategies, but they are mostly expected to finish a problem, check its correctness and move on to the next problem.Hensberry and Jacobbe (2012), Hiebert (2003) and Schoenfeld (1987) declare this underrepresented reflective part of the problem-solving process to be crucial, as the discussion of solutions and how they have arrived at them allows students to explain their thinking, learn from other students, and develop more efficient methods for solving problems.

In Wilson et al. (1993), Wilson reports that he conducted a year long in-service mathematics problem solving course for secondary teachers. At the end of the year, one of the teachers observed: „In schools, there is no looking back." and the participants agreed that getting students to engage in looking back activities was difficult. Wilson comments that reasons cited in this context were entrenched beliefs that problem solving in mathematics is answer getting, pressure to cover a prescribed course syllabus, testing (or the absence of tests that measure processes), and student frustration.

Pólya (1945, p.14) observed similar problem-solving behaviour with his students: „Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work." Pólya (1945, p.15) explains how a good teacher should behave in this context: „A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted."

Apart from these rare observations there are only few investigations how teachers make their students engage in looking back activities in the classroom during problem-solving classes.

**Empirical studies on the phase looking back in the classroom**

In their explorative empirical study, Heinrich, Pawlitzki and Schuck (2013) studied 16 problem-solving lessons at primary school level. They subdivided the observed lessons into three phases: (1) Presentation of the problem and understanding the problem, (2) Devising and carrying out a plan, (3) Presenting and discussing the solution(s).

Heinrich et al. (2013) identified the following four elements during phase (3) within the observed lessons:

- Revising the solutions
- Dealing with alternative solutions
- Dealing with difficulties and mistakes
- Follow-up problems, similar strands of work

The data of the study shows that on average each of these above measures was witnessed (only) in about one fourth to one third of the lessons.

Another empirical study that focusses on the teachers' behaviour during the phase looking back was accomplished by Leong et al. (2011). The aim of this study was to determine how two teachers who have undergone a substantial professional development programme think about and carry out the phase looking back in their instructional practice.

The evaluation of the data showed that both of the two teachers, whohad had special training concerning each of Pólya's phases, were willing to commit relatively large amounts of time in their lessons to the phase looking back. The way they carried out the phase looking back in their lessons indicates that they went beyond the usual notion of „checking for answers" and closer to the ideals of „extension", „alternative solutions," and „generalisations" that were envisioned by Pólya (1945) (see Leong et al., 2011).

Leong et al. (2011) and Heinrich et al. (2013) both lament the scarcity of research on the phase looking back in the classroom.

**Research questions**

The following research questions should be answered through the empirical study:

- How do teachers organise the phase looking back? Do the teachers implement elements of the phase looking back that were envisioned by experts like Pólya, Schoenfeld, etc.? Do the teachers discuss further elements during the phase looking back with their students?
- Are there any profiles of the teachers' behaviour?

**Data and analysis**

In order to answer the above questions, an empirical study was designed[12] in which 14 mathematics lessons (each 90 mins) in grades 9/10 in German secondary schools were videotaped. Two weeks before the lesson, the participating teachers received a geometrical problem that was supposed to be dealt with in the lesson, including possible solutions. The participating teachers, who had received no special training beforehand with regard to Pólya's phases, were supposed to hand in a lesson plan before the lesson. There were three interviews with the participating teachers in which they could reflect on the actual lesson and on mathematical problem-solving in general: before, immediately after and some days after the lesson.

Transcriptions of the final phase in the lesson, in which the solutions werepresented and discussed,were prepared, as well as transcriptions of the interviews. In order to analyse the data, a category system on the basis of the categorization of the phase „Review" by Mason et al. (1982) was set up. It was supplemented by elements of the system for categorizing metacognitive activities by Cohors-Fresenborg andKaune (2007). In addition to this theory-guided approach, the category system was complemented by evidence-based elements from the available data. This category system is supposed to characterize the teachers' actions during the phase looking back.

An initial analysis shows that indeed many teachers rather focus on „checking that the solution is correct" while neglecting further reflections on the problem.

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[12]Supported by Stifterverband für die Deutsche Wissenschaftand Volkswagen AG

Furthermore, extensions of the geometrical problem were very rarely observed, even rarer than further reflections on the problem.The analysis of the data is still in progress, but the first results already indicate that teacher instruction should pay (much) more attention the complex notion of the phase looking back as it was envisioned by Pólya (1945).

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# TYPES OF CONTROL IN COLLABORATIVE PROBLEM SOLVING

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*In this paper, the types of social metacognitive control actions during collaborative problem solving are examined. Six pairs of future elementary school teachers work on two problems, an arithmetical and a geometrical one. The analysis of the collected data gave evidence of two types of control actions, Global and Context-based ones. At the same time, the factors that trigger control actions in a collaborative problem solving were also examined. It was found that some of these factors are a multiple solution task, an unreasonable result, the mismatching between the outcome of the problem-solving and verification processes, and finally the collaborative nature of the problem-solving.*

## INTRODUCTION

Performance on many tasks is positively correlated with the degree of one's metaknowledge (Schoenfeld, 1985). Executive control and decision making is a way to talk about metacognition in problem-solving. Both are concerned with the solver's continuous evaluation of her/his current working in conjunction with her/his aim. This means that the solvers monitor and assess both the state of their knowledge and the state of the solution and avoid „wild chases" that often guarantee failure in the evolution of the solution (Schoenfeld, 1985,1988). The solver's focus is on decisions about what to do in order to solve a problem. „Behaviors of interest include making plans, selecting goals and subgoals, monitoring and assessing solutions as they evolve, and revising or abandoning plans when the assessments indicate that such actions should be taken" (Schoenfeld, 1985, p. 27).

The vast majority of the relevant studies emphasize control actions taken by individuals. The study of Stillman and Galbraith (1998) is a nice example. In their study they found that the students exhibited knowledge of strategies (i) to aid understanding of the problem (such as re-read the problem and relate the problem to other problems within experience), (ii) to organize information (they were aware that organizing information into a table facilitates the process), (iii) for developing and executing plans, (iv) for monitoring process, and (v) for verification of the final results.

However, recent studies (Chiu & Kuo, 2009; Chiu, Jones & Jones, 2013) raise the importance of social metacognitive control during collaborative problem-solving. In this setting of collaborative problem-solving we examine the kinds of control actions the solvers take in their effort to solve a problem as well as the factors that trigger these control actions. Therefore, our research questions are:

- What control actions can be identified during collaborative problem-solving?
- What factors trigger control actions during collaborative problem-solving?

## SOCIAL METACOGNITIVE CONTROL

According to Chiu et al. (2013) social metacognitive control refers to „monitoring and control of the regulation and evaluation of others' knowledge", linking thus metacognitive judgments with communication, and applying one's subjective metacognitive experiences to a group context (p. 74). In addition to the traditional components of individual metacognitive control, social metacognitive control includes group interaction and social influence. When mathematics problems are solved by groups rather than individuals, team members who can monitor and control other's behaviors effectively, can increase their likelihood of solving difficult problems. This is a vital issue and it benefits both the problem-solving procedure and the solvers. On the one hand, this control can ensure that the answer is mathematically accurate and reasonable (Goos & Galbraith, 1996). On the other hand, it helps solvers to develop self-reflection, mental thinking and become capable solvers (Wilson, Fernandez & Hadaway, 1993). Chiu et al. (2013) following two students during their attempt to solve a problem found that the use of metacognitive control strategies made them able to understand, evaluate and build on each other's thinking respectfully, creating thus new useful information that broadened their mathematics problem-solving. Through social metacognitive control, the cognitive and metacognitive demand that is necessary for solving a complex problem, is distributed among the group members, allowing the solvers to focus on smaller and simpler responsibilities according to their skills (Chiu & Pawlikowski, 2013). Nelson, Kruglanski and Jost (1998) found that as the group members model for one another, trying to understand and solve a problem, they actually help them interpret it and make effective metacognitive control decisions. Collaborative work however, might have also some limitations. To obtain effective social metacognitive control, sufficient communications skills are necessary (Kaiser, Cai, Hancock & Foster, 2001). Therefore, poor communication skills might influence performance and result to ineffective feedback (Efklides, 2009). Moreover, limited individual content knowledge might also result to inappropriate feedback that could mislead the group members (Holton & Thomas, 2001).

## DESIGN OF THE STUDY

Twelve undergraduate students of the Department of Primary Education in the Aristotle University of Thessaloniki participated voluntarily in this study. The students' mathematical background was weak despite they attended two optional courses (Elementary Mathematics and Problem-solving) plus the obligatory one on

Mathematics Education. They were asked to form pairs based on their personal preferences in order to ensure efficient co-operation.

Two problems were given to each pair, an arithmetical and a geometrical one. They had in total thirty minutes to solve each problem.

This is the arithmetical problem:

*„If we line up the school's students in rows of three, two of them are left. If we place them in rows of four or five, two of them are also left. What is the number of students, given that this is a 3-digit number and the sum of its digits is 5?"*

This is not a difficult problem. But, it is interesting for two reasons. First, its solution combines a variety of mathematical concepts and/or mathematical ideas (use of Lowest Common Multiple and its multiples, use of the divisibility rules, all the possible combination of three digits that give a sum of five, etc.). Second, it has more than one possible solution. Initially there are multiple numbers that satisfy the first three conditions (2 students are left when in rows of 3, 4, or 5). But, the number of the possible solutions is reduced to 2 solutions due to the fourth condition (sum of digits is 5). The two solutions are 122 and 302.

This is the geometrical problem:

*„Find the area of the shape"* (Fig. 1)

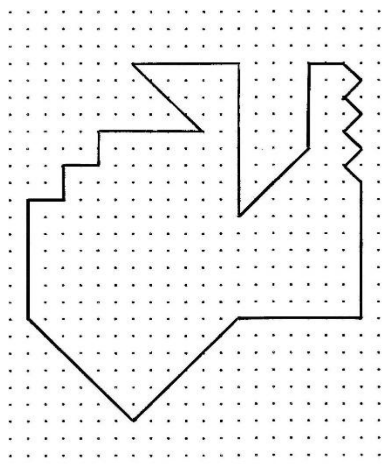

Figure 1. The shape of the geometrical problem

This is also a typical geometrical problem related to the concept of area which is easy for an experienced solver. There is only one correct answer (245,5 square units) but this answer can be approached in various ways. For example, one could use Pick's formula to calculate the area A(P) of any lattice polygon: A(P)=I+B/2-1 ('I' denotes the points in the interior of the polygon whereas 'B' the points on its boundary). Another approach is the division of the shape to sub-shapes. This means that the solver must find the area of each sub-shape and then, their sum makes the area of the whole shape. An alternative could be to circumscribe a rectangle around the polygon, calculate its area, then find the area of the shapes that exist between the initial shape and the rectangle, add their areas, and subtract their sum from the area of the rectangle. The fact that both problems are open problems, amenable to multiple solutions/strategies (Silver, 1997) is a factor that triggers control actions. As Cifarelli and Cai (2005) claim, in open ended situations the solvers continually monitor and modify their problem-solving actions.

The students were also asked to vocalize their thoughts while performing the tasks. The think-aloud strategy refers to having students verbalize what they are thinking and doing throughout the problem-solving session. Students who verbalize seem to gain a better understanding of their own thinking (Albert, Bowen & Tansey, 2009) and therefore are prepared to ask appropriate questions to enhance their understanding of the problem-solving process. They also seem better able to trouble-shoot their own problems, and in many cases, solve those problems on their own (Pugalee, 2001). These make the „think aloud" another factor that triggers control actions during problem-solving.

The whole procedure was audio-recorded and then the students' efforts were transcribed for the purpose of the paper. Their worksheets and the transcribed protocols constituted our data. The students' protocols were parsed into episodes (according to Schoenfeld's (1985) proposed Framework for Analysis of Problem-Solving Protocols), giving emphasis on the points showing aspects of control and on instances indicating that certain factors trigger control actions.

## RESULTS AND DISCUSSION

The control actions identified in the analyzed protocols were finally organized in two collections according to whether these actions dependent on the task's context or not. The first category is called „Global control actions" (not depended on the task). It is about control actions that can be applied to any problem no matter the content/mathematical topic of the problem (Fig. 2). This makes an analogy to the notion of heuristics. The second category is called „Context-based actions" (task-dependent). These actions are closely related to the context of the specific task. It is the nature of the task, its content, that provokes these specific control actions. These actions can be transferred only to similar tasks. In the next section, each reference to excerpts from the transcribed protocols is accompanied with an alphanumeric part on the left that needs to be explained in advance. This includes the number of lines of the protocol (e.g., 27-29 or 130). Then, a combination of letters and numbers is used to indicate the number of the pair, the number of the student, and the kind of the problem. For example, P3.1.A means Pair-3, Student-1, Arithmetical problem (and similarly G for the geometrical one).

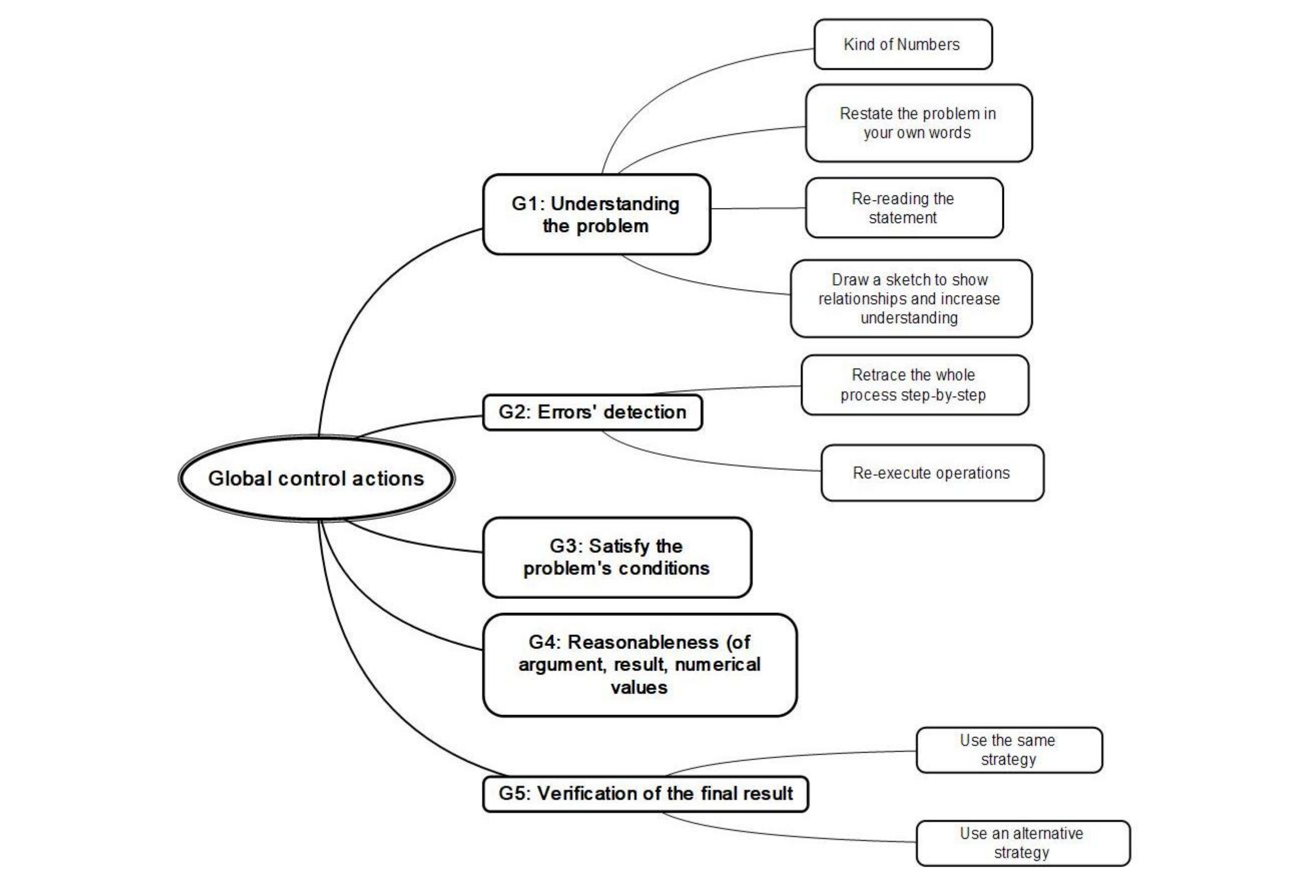


Figure 2. Global Control Actions

**Global control actions.**

***G1 – Understanding the problem***

This refers to actions that aim to control the correct understanding of the problem. The collected data allowed us to identify several types that fit to this global control action. In some cases, the solvers tried to check their understanding by using the principle „Check the kind of the involved numbers to see whether you understand the problem" (G1). This means that they connected the correct understanding of the problem with the identification of the kind of numbers that might be (or not) a possible solution. For example, pair-3, in their early discussions while trying to correctly understand the arithmetical problem, made clear that any solution including decimal numbers must be considered as unacceptable. The problem asks for the number of students and therefore the answer must be a positive integer.

27-29 P3.1.A: We speak for students, so … We are not allowed to find decimal numbers.

Another principle in this global action was: „Restate the problem in your own words". This refers to the well-known suggestion of Polya (1973) that „the verbal statement of the problem must be understood" (p.6). The solver "should be able to state the problem fluently”. This was a quite frequent instance of control action in our study. Pair-4, in their effort to understand the arithmetical problem restated it in this way:

17-18 P4.1.A: The total number of students in a primary school is not a multiple of 3, 4, and 5 since each time we line up the students in rows of 3, 4, and 5, two of them are left.

The third way to ensure the correct understanding was to „Re-read the statement of the problem from time-to-time”. All the pairs took advantage of this action while solving both problems. When pair-3 got two answers for the arithmetical problem there was a doubt whether they understood correctly the problem:

130 P3.1.A: So, we made a mistake. We should find only one answer.

131 P3.2.A: Not necessarily. Let’s re-read the problem to check whether there is a clue for the existence of a unique answer.

Finally, another way to check the correct understanding was to „Draw a sketch or diagram to show connections and relationships to increase understanding”. There is an initial understanding and then a sketch is used to confirm or challenge this understanding. Trying to solve the arithmetical problem, one member of pair-5 drew a simple sketch to make sure that both solvers understand in the same way the problem (Fig. 3).

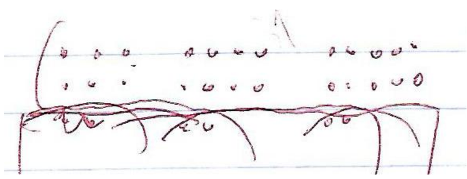

Figure 3. Draw a sketch to check your understanding

### *G2 – Errors’ detection*

This type refers to the solvers’ efforts to check for the existence of errors in the intermediate results (thus not connected with the issue of verifying the final result) during the problem-solving process. Two types were identified in this global control action.

The first one is based on „Retracing the whole process step-by-step”. In this case, the solvers retrace the whole problem-solving process step by step and check each one of their steps to make sure they didn’t make any mistake. For instance, pair-3, decided to use Pick’s formula but they did not remember it. So, they decided to re-invent the formula. They worked on a series of lattice polygons and finally they came up with ‘the formula’. Then, they wanted to examine whether this formula is the correct one. So, they followed the same series of steps, using the same shapes, in the same order, which actually helped them to identify an error and correct the formula.

The second one is the „Re-execution of arithmetical operations". The aim of the solvers was to ensure that the intermediate calculations were correct. For example, a member of pair-2 explained to his partner:

107 P2.2.G: Are you sure?

108 P2.1.G: Yes! It's okay to find a decimal number

108 P2.1.G: 2 + 3 = 5, 5 x 1 = 5, 5 / 2 = 2,5.

110 P2.1.G: I just did it again to see whether we made a mistake.

***G3 – Satisfy the problem's conditions***

The aim of this control action was to check whether the problems' conditions are satisfied. The solvers decide their next step based on whether this decision is in alignment with the problem's conditions. One of the conditions in the arithmetical task was that the answer must be a 3-digit number. Pair-1 examined whether certain numbers would be valid choices for the first digit. Based on the above-mentioned condition they decided that:

85 P1.1.A: The first digit cannot be zero….

For the same problem, pair-4, based on the condition that if students are lined up in rows of five, two are left over, decided that:

153-154 P1.2.A: Neither zero nor five are proper options for the last digit

***G4 – Check the reasonableness***

This action concerns the students' engagement in checking the appropriateness and reasonableness or accuracy of their current work (argument, intermediate result, numerical values, measurements). The Common Core State Standards (National Governors Association, & Council of Chief State School Officers, 2010) give emphasis on the necessity for students being able to „Reflect on the reasonableness of their answers", and „self-assess to see whether a strategy makes sense as they work, checking for reasonableness prior to getting the answer". The relevant examples are drawn from the geometrical problem concerning the area of an irregular shape. Pair-3, trying to find the area of the shape got -199 square units as their final result. Checking the reasonableness of their result they soon realized that this was incorrect:

152-153 P3.2.G: The area is minus 199 square units. What have we done? This is a negative area.

In the same spirit, pair-6, in order to calculate the area of the irregular shape, circumscribed a rectangle. Then, they found the area of each sub-shape that exists between the initial shape and the rectangle, added these areas, in order to subtract their sum from the area of the rectangle (Fig. 4). However, according to their calculations, the total area of the irregular shapes within the rectangle was bigger than the area of the whole rectangle. Checking the reasonableness of their result they were able to realize that they made a mistake:

106-108 P6.2.G: The area of the shapes within the rectangle is bigger than the area of the rectangle. This is not reasonable.

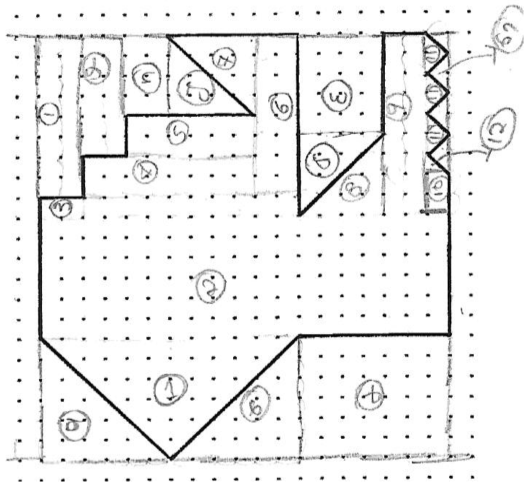

Figure 4. The circumscription of a rectangle outside of the irregular shape

### *G5 – Verification*

According to Eizenberg and Zaslavsky (2004) „verification is one aspect of metacognitive control skills" and is connected with Polya's 'Looking back' final stage addressing the question „Can you check the result?". So, to check the final result the participating pairs followed two different approaches.

The first one was to „solve again the problem using the same strategy". The students repeated the **same** steps in the **same** order using the **same** problem-solving strategy. If this leads to the same answer, then the answer is correct. Otherwise, each step should be checked again separately. Pair-1 used the Pick's formula to calculate the area of the irregular shape and then they wanted to verify the result. Their decision was:

234-235 P1.1.A: Let's do the same again. Let's count the dots one more time please.

The second approach was to „solve again the problem using an alternative strategy". In this case the calculated result was verified through the use of an alternative strategy for solving the problem. If this alternative strategy leads to the same result then the answer is correct. So, the students apply strategically two problem-solving paths. The first one for solving the problem and the second for verification purposes. Pair-4 decided to use initially the Pick's formula to calculate the area of the irregular shape. Then, in order to verify the result, they decided to divide the shape to sub-shapes. They calculated the partial areas of these shapes and checked their initial result through the sum of these partial areas:

115-119 P4.2.G: This second method was the verification. So, the area is 245,5 square units… We measured the area of the smaller shapes, added them and got the same number.

**Context-based control actions**

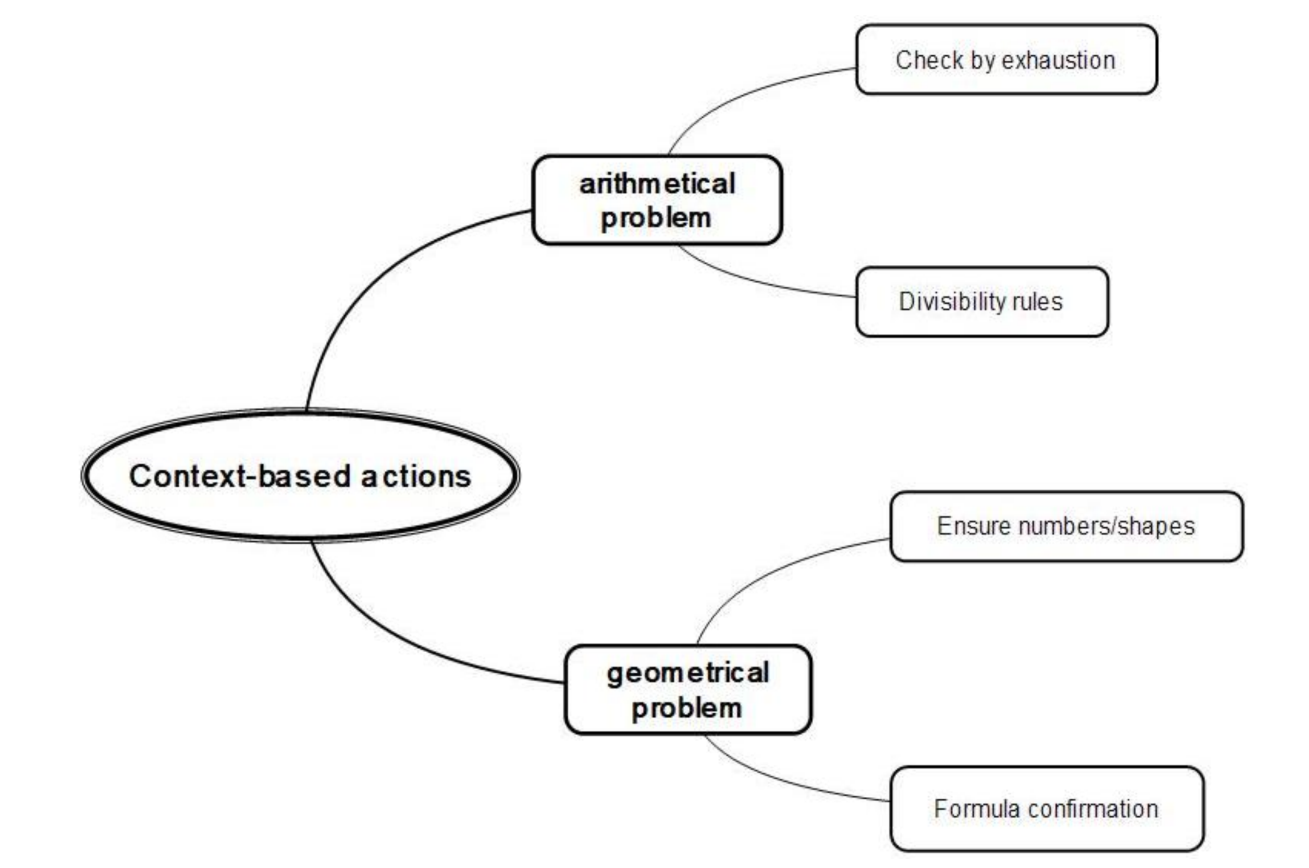


Figure 5. Context-based control actions

These actions are exclusively connected to the specific problem since it is the content of the task that provokes them. Consequently, such local actions were identified separately for each problem (Fig. 5). More specifically, two types of actions were identified for each problem.

***Arithmetical problem***

***CA1 – Systematically exhaust cases***

The first control action for the arithmetical problem is to systematically exhaust all the possible cases. Based on one of the problem's conditions the pairs wanted to form all the possible triads of digits that have the sum equal to 5. This could be done by forming these triads in a random way asking only for the sum of the digits. However, such an approach cannot guarantee that the solver will be able to find all the possible triads. Therefore, the question for the pairs was to find a way of systematically working on triads so as to make sure they did not miss any triad. This made them move towards a systematic way of controlling the way the digits can be combined. So, they decided to start with the smallest digit (i.e., 1) in the first place (Fig.6, left). They kept it unaltered asking for the remaining two digits that would have the sum equal to 4 (so as to meet the demand for three digits that have the sum equal to 5). Then, the next choice for the first place was number 2, and so on. This resulted to a list of all the possible triads. Some pairs chose to work on the reverse order, starting with the biggest possible digit in the first place (i.e., 5) (Fig.6, right). This also ensured that all the possible triads were considered. The main aim of this control action was to check that all the possible triads have been examined.

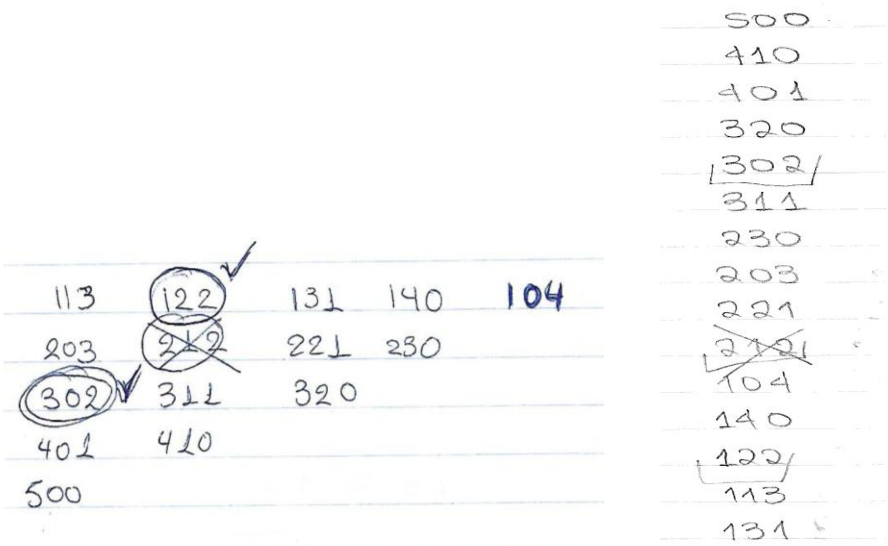


Figure 6. Systematic working using a list

***CA2 – Divisibility rules***

The second context-based control action for the arithmetical problem was the use of the divisibility rules in order to check possible solutions. These rules determined the decision to keep or abandon a possible solution. This can be seen clearly in the discussion between the members of pair-4. They considered number 104 as one of the possible solutions for the arithmetical problem:

| | | |
|---|---|---|
| 48-49 | P4.1.A: | It could be 104. The sum of its digits is 5. |
| 50-51 | P4.2.A: | Yes. But, the last two digits form a 2-digit number that is divisible by 4, which means that the whole number is divided by 4. However, the number we ask for cannot be a multiple of 4 |
| 52 | P4.1.A: | You are right |

### *Geometrical problem*

### *CG1 – Ensure numbers/shapes*

For the geometrical problem, the first task-dependent control action was to ensure numbers or shapes. The use of the Pick's formula needs both the number of the lattice points that are on the boundary of the shape and those that are in its interior. The solvers had to make sure that all these points had been considered. This refers to actions aiming to control that all the boundary or interior points have been considered without missing or duplicating any of them. Therefore, the participating pairs took actions such as the use of their own marks (circles, lines, different colors, and numbers) to indicate that a specific point had been considered (Fig. 7).

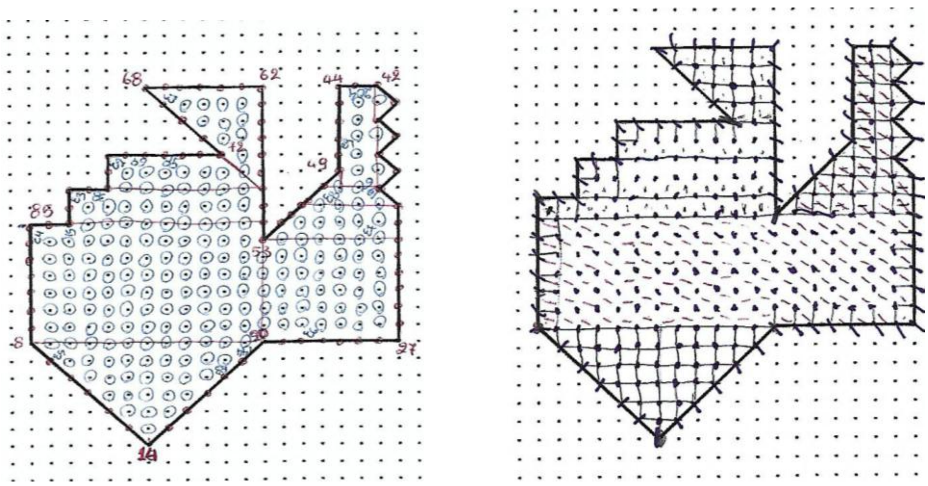

Figure 7. Ensure numbers

In the same spirit, when the solution was based on the division of the initial shape to sub-shapes, the pairs had to be sure that they considered all the participating shapes for calculating the total area also without missing or duplicating any shape. In this case they used numbers to indicate that the specific sub-shape had been considered (Fig. 8). This use of numbers in their worksheets constitutes a control action.

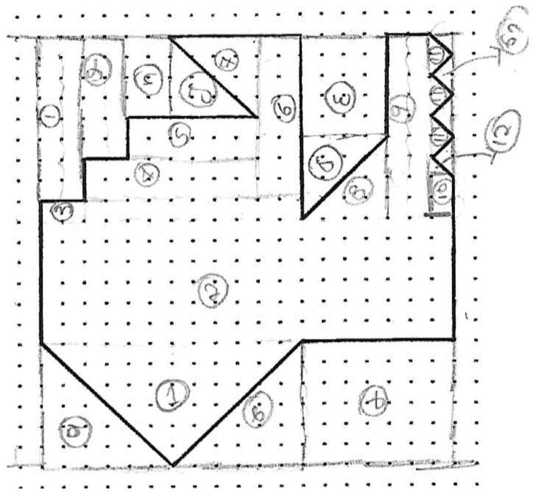

Figure 8. Ensure shapes

***CG2 – Formula confirmation***

This control action is connected with the solvers' aim to re-discover the Pick's formula because they wanted to use it, but they did not remember it. So, their first step was to find the formula. This was not an easy step. They examined a series of simple shapes to extract the formula (Fig. 9).

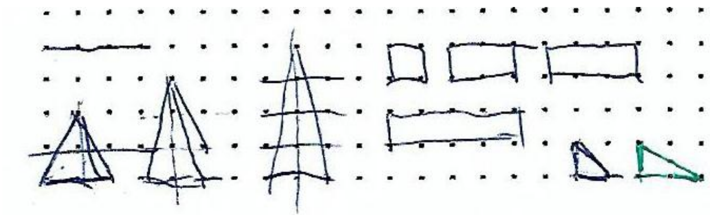

Figure 9. Discovering Pick's formula

This was a process of problem-solving within problem-solving. They left aside the initial problem to work with this new one. Their effort resulted to a formula and then they wanted to check whether this formula was the correct one. They decided to use random shapes to check the strength of their formula (Fig. 10).

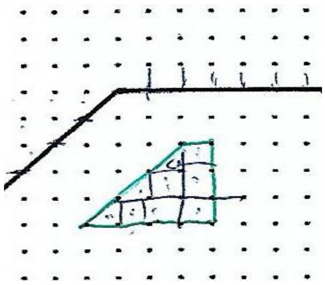

Figure 10. Checking the formula

They were able to calculate the area of these shapes (for example using the grid). Then they applied their formula for the same shape. If the two results were identical, this meant that the formula was correct. This decision to use random shapes to check the validity of the formula was actually a control action at the local level of the geometrical problem. This is evident in the work of pair-3:

| | | |
|---|---|---|
| 226 | P3.2.G: | If the formula is correct, it means that we can find the area of any shape. |
| 227 | P3.1.G: | Yes |
| 228 | P3.2.G: | Make a random shape quickly |

**What triggers control actions?**

The analysis of the transcribed protocols gave evidence about the possibility for some factors that trigger control actions. We were able to identify four of them:

- The existence of multiple solutions.
- An unreasonable result
- Both problem-solving and verification processes lead to different answers.
- The collaborative nature of problem-solving.

***The existence of multiple solutions***

The successful solution of the arithmetical problem results in two solutions. The required number of students is 122 or 302. Both are correct answers since they satisfy all the conditions of the problem. However, it was not easy for most of the pairs to accept two solutions for the same problem. The extracts below are representative of this difficulty:

| | | |
|---|---|---|
| 130 | P3.1.A: | We rather made a mistake since we had to find only one answer. |
| 129 | P5.1.A: | Now we must find out which one (of the two) is the correct answer |
| 93 | P6.2.A: | If we could have only one number our solution would be complete |

This difficulty in accepting more than one solutions made them either looking back again to their problem-solving path to check for possible mistakes or using an alternative approach to see whether their multiple answers make sense.

***An unreasonable result***

Another factor that triggered control actions was the emergence of an unreasonable result. The negative area in the geometrical problem as already mentioned earlier was an example of an unreasonable result that triggered control actions. Another example was the result 100.25 for the population of the school in the arithmetical problem. The answer to this problem cannot be a decimal number. By realizing the unreasonableness of this result the students turned towards checking their argumentation to see what was the reason that misled them.

| | | |
|---|---|---|
| 119-120 | P1.1.A: | We cannot have 100 point something… when talking about the number of students. |

***Problem-solving vs. verification***

The third factor that appeared to trigger control actions was when the problem-solving and verification processes resulted in different answers. In this case, the solvers were led to check their results to resolve the discrepancy. This was noticed in two slightly different ways as it has already been mentioned in the subsection about verification. In the first, the solvers used the same method twice. Initially to solve the problem, and then they followed the very same process step-by-step to check the correctness of the result. In the second, the solvers chose two different approaches, one to solve the problem and one to check the correctness of the result. In both cases, any disagreement between the two results was the reason for undertaking control actions to decide how to proceed.

### *The collaborative nature of problem-solving*

Perhaps the most powerful factor that triggered control actions was the collaborative problem-solving in itself. During this collaborative work „partners attend to both their own and the other's knowledge, actions and emotions through invitational questions, evaluations, and responses" (Chiu, Jones & Jones, 2013). Usually, one of the members suggested the next step. Then, his/her partner was wondering why this was the best choice, or whether this was the right step. So, questions, such as „What do you think?", „Is that right?" arose. These questions aimed to monitor and control their individual effort to solve the problem as well as to make explicit their mutual understanding, to make each one participate and pay attention to the other's actions and thoughts. In the 130.P3.1.A-131.P3.2.A extract it was the objection of the second student („Not necessarilly") and his prompt („Let's re-read the problem to check whether there is a clue") that led to this specific action. In this case a proposal („We made a mistake. We should find only one answer") was aborted due to the doubt of the co-solver. Similarly, in the 234-235. P4.2.G excerpt, the pair repeated the same steps in the same order to ensure that the result is correct. This control action was the consequence of the prompt made by one of the students („Let's do the same again. Let's count the dots one more time please"). So, in this case a proposal for a control action was initiated by one member of the pair and was approved by the other.

## CONCLUSION

We know that metacognitive control has been mainly, but not exclusively, examined from the perspective of the individual, rather than the collective one. Examining control actions during collaboratively problem solving, gave evidence about the way pairs divided complicated tasks into sub-tasks, allocated these sub-tasks appropriately in order to distribute the demand in cognitive and metacognitive level. For example, in their effort to solve the given problem they posed a new one, i.e., the problem of finding the Pick's formula which was a necessary tool for their solving approach. The social metacognitive control sometimes increased the transparency of the used argumentation and this is why we made an effort to distinguish two general types of such control actions. On the one hand, there are control actions that have a universal character and can be applied potentially to any problem. On the other hand, there are actions connected exclusively with the specific tasks and cannot be applied in other tasks, unless they are similar to the given one. In addition to these, we observed four certain factors that seem to trigger control actions. These are the existence of multiple solution for a problem, the emergence of an unreasonable result, the case that the problem-solving process and the verification result in different answers and the collaborative problem solving itself.

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# PROBLEM SOLVING IN THE CLASSROOM: HOW DO TEACHERS ORGANIZE LESSONS WITH THE SUBJECT PROBLEM SOLVING?

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*In this article, a theoretical grid to describe the behaviour of teachers during problem solving lessons, based on Pólya's problem solving phases and the teachers' handling of problem solving strategies, is introduced. Furthermore, the behaviour of several teachers is interpreted in connection with their beliefs regarding the nature of mathematics and especially problem solving, revealing great consistency.*

## INTRODUCTION

This article focuses on two important aspects that concern teaching mathematics and especially teaching problem solving in schools: (1) teacher behavior and (2) teachers' beliefs. Each of these two aspects is addressed with research questions and an additional research question concerns the interaction of both aspects.

*(1) How do teachers organize lessons with the subject „problem solving"? How can their behavior best be characterized?*

*(2) What are teachers' beliefs regarding problem solving and the nature of mathematics? How can these beliefs be captured?*

*(3) Is the behavior of teachers related to their beliefs regarding problem solving and the nature of mathematics?*

The article at hand gives an insight into the exploratory study „Problem Solving in the Classroom" (ProKlaR – Problemlösen im Klassenraum) which is still ongoing. The research questions above have already been addressed in Rott (2016), and the answer to question (2) will be the same as in the previous article: interviews and mathematical views by Ernest (1989a) are used to gather and describe according data. Research question (1), however, will be answered in a new way, leading to a more meaningful answer to questions (3) compared to previous publications of this project.

The results presented below relate to the behavior and beliefs of three teachers that will be discussed in detail. In the discussion part of this article, data of seven teachers and their lessons, respectively, will be used to answer research question (3).

## THEORETICAL BACKGROUND

This chapter, following the outline presented in the introduction, is divided into three sub-chapters: (1) teacher behavior, (2) teachers' beliefs, and (3) the interplay of teachers' behavior and their beliefs.

### (1) Teacher behavior

In the literature, different types of behavior (types of teaching) that are typical are reported. Mostly, the reported patterns of action are unfavorable and suggestions are made to improve that kind of behavior. One goal of the study „problem solving in the classroom" is to find such kind of typical behavior and describe it in more detail. The review revealed the following types:

(a) There are teachers that make sure that all their students understand the problem before they start working on it (e.g., Knoll, 1998; Blömeke et al., 2006). This way, they take the responsibility for the „understanding of the problem" phase and their students cannot learn this.

(b) There are teachers that split problems into „small bits" for their students and trivialize it. This way, no real problem solving is possible (e.g., Schoenfeld, 1988).

(c) There are teachers that prefer their own solution and do not appreciate the ideas of their students (e.g., Knoll 1998).

(d) There are teachers that (in phases of looking back) focus solely on the results without talking about strategies or the lack thereof (e.g., Heinrich et al., 2013).

(e) There are teachers that give content-related aid that directly lead to a solution very early without trying motivational or strategical aids beforehand (e.g., Zech, 1998).

(f) There are teachers that try to give as few feedback as possible to not influence their students' individual ideas (Rott, 2015).

Sometimes, these types of teaching and typical behaviors refer to a whole lesson, and sometimes they refer to specific phases of lessons or the problem solving part of lessons, respectively. For example, pattern (a) refers directly to the „understand the problem" phase and to no other phases of a lesson, whereas pattern (d) solely refers to the „look back" phase. On the other hand, pattern (c) might refer to different phases of a problem solving process as focusing on one approach might affect planning as well as implementing and looking back.

To be able to classify and work with these patterns, I am going to introduce a grid derived by theoretical considerations. I have to admit that it took me some time to develop the grid, even though, in retrospect, it seems to be obvious. As pointed out above, the behaviors refer to different parts of problem solving processes which can be described by Pólya's (1945) four phases. For each of those phases, the teacher's actions can be neutral (not narrowing their students approaches but also not emphasizing strategic diversity, either), closely managed (preferring one approach that the students are being lead to), or emphasizing strategies (encouraging the students to pursue their ideas). This grid is elaborated and the different cells of the grid are operationalized in Table 1.

As pointed out above, the typical behaviors from the literature often address only single cells from this grid. To fully characterize teachers' behavior in their lessons, one cell in each row should be coded resulting in „profile lines" for teachers' actions in lessons with the subject problem solving (see methodology).

| Phase | closely managed | neutral | emphasizing strategies |
|---|---|---|---|
| **Understand the problem** | The teacher explains the problem (formulation). | The teacher does not comment on the problems or answers questions. | The teacher gives hints, but does not explain the problem. |
| **Devise a plan** | The teacher tells the students which („correct") approach to use. | The teacher does not give any guidelines and strategic support. | The teacher hints at [ideally different] approaches and encourages the students to follow their ideas. |
| **Carry out the plan** | The teacher gives the students concrete content-related support (early). | The teacher gives (almost) no (strategic) help. | The teacher gives staggered aids (motivational / feedback / general strategic / task-specific strategic / content-related). |
| **Look back** | Fixation on the result; perhaps, one (arithmetic) approach is presented. | Different approaches are presented; however, strategic ideas or the differences between approaches are not highlighted explicitly. | The approaches and strategies are highlighted; the result might also be presented but is of secondary importance. |

Table 1: Different options for teachers to organize phases in problem solving lessons.

### (2) Teachers' beliefs

At the end of the 1980s, Paul Ernest (1989a) summarized his research on teachers' beliefs, stating that there are three beliefs regarding the nature of mathematics: the *instrumentalist view*, the *Platonist view*, and the *problem-solving view*.

> First of all, there is the instrumentalist view that mathematics is an accumulation of facts, rules, and skills to be used in the pursuance of some external end. […] Secondly, there is the Platonist view of mathematics as a static but unified body of certain knowledge. […] Thirdly, there is the problem-solving view of mathematics as a dynamic, continually expanding field of human creation and invention, cultural product. (Ernest, 1989a, p. 250)

Even earlier, Dionne (1984) published a questionnaire to evaluate the perception of mathematics that teachers have. He differentiated between three different perceptions: (1) the *traditional perception*, mathematics as a *collection of rules* (doing mathematics is doing calculations, using rules, using procedures, using formulas). (2) The *formalist perception*, mathematics as the *science of formal* *structures and of rigorous logic* (doing mathematics is writing rigorous proofs, using a precise and rigorous language, using unifying concepts). And (3) the *constructivist perception*, mathematics as *something to do and not only to look at*, one gets a piece of knowledge if and only if one has reconstructed it for oneself (doing mathematics is developing thinking processes, building rules and formulas from experiences on reality, finding relations between different notions).

In Germany, Günter Törner and his colleagues (Grigutsch et al., 1998) published one of the first big studies regarding the beliefs of teachers. They identified two fundamental ideas about mathematics: mathematics as a static *system* or as a dynamic *process*. They add that both ideas cannot be separated from one another, often referred to as a „Janus-faced" characteristic of mathematics.

In their questionnaire to identify teachers' beliefs, Grigutsch et al. distinguished the „system" idea into „formalism" and „schema"; they also added „application" as another dimension, resulting in four belief dimensions: *process*, *schema*, *formalism*, and *application*. Those four dimensions have been confirmed empirically in a study with 310 in-service teachers that filled out the questionnaire.

A comparison of the belief systems by Ernest, Dionne, and Törner reveals a good fit between those three conceptualizations: the *problem-solving view* is equivalent to the „constructivist perception" and „mathematics as a process". The *Platonist view* of mathematics can be matched to the „formalist perception" and „mathematics as a system". Moreover, the *instrumentalist view* of mathematics (sometimes referred to as „mathematics as a tool box") is addressed by the „traditional perception" and the „application" aspect of mathematics (see also Pehkonen, 1999).

In the rest of this article, only the category names by Ernest will be used to speak of teachers' beliefs.

## (3) The interplay of teachers' behavior and beliefs

Beliefs are often characterized as „psychologically held understandings, premises, or propositions about the world that are thought to be true" (Philipp, 2007, p. 259): They can be held consciously or unconsciously, either way – like lenses – they influence the way we see the world and our actions (ibid.).

Several studies document the influence of beliefs related to the nature of mathematics and/or problem solving, especially beliefs held by students (which will not be addressed in this article) but also beliefs held by teachers. For example, Schoenfeld (1992) summarizes studies indicating that teachers' beliefs on problem solving (in the classroom) are often related to the teachers' underlying conceptualizations of (the nature of) mathematics. This is in accordance with Fennema and Franke (1992) who report that teachers' conceptions of subject matter and their epistemological beliefs play an important role in shaping the teachers' classroom practice. The same is true for Beswick (2012) who shows that it is possible for teachers to hold differing beliefs regarding mathematics as a school subject and mathematics as a scientific discipline and that the latter can be important to understand the teachers' actions.

Consequently, Ernest describes how the three different beliefs (or views, respectively) might influence teachers' behaviors in their classroom practices in general:

> For example, the instrumental view of mathematics is likely to be associated with the instructor model of teaching, and the strict following of a text or scheme. It is also likely to be associated with the child's compliant behaviour and mastery of skills. Similar links can be made between other views and models, for example: Mathematics as a Platonist unified body of knowledge – the teacher as explainer – learning as the reception of knowledge; Mathematics as problem-solving – the teacher as facilitator – learning as the active construction of understanding, possibly even autonomous problem-posing and problem-solving. (Ernest, 1989a, p. 251 f.)

In another publication, Ernest even specifies the influence of the three different beliefs on teaching problem solving:

> The different philosophies of mathematics have practical classroom outcomes. For example, an active, problem-solving view of mathematical knowledge can lead to the acceptance of children's methods and approaches to tasks. In contrast, a static Platonist or instrumentalist view of mathematics can lead to the teacher's insistence on there being a single 'correct' method for solving each problem (Ernest, 1989b, p. 21)

In the study at hand, the hypotheses by Ernest regarding the influence of beliefs on actions in the classroom will be examined.

## METHODOLOGICAL DECISIONS

To analyze (1) teachers' behavior in the classroom, they have to be observed during teaching. Therefore, only teachers that agreed to being filmed during their lessons, were included in this study, resulting in a selective sample. The lessons were videotaped using one camera at the back of the room, focusing on the blackboard and the teacher (during plenary phases).

To ensure the topic problem solving, the participating teachers were asked to conduct a lesson based on one (or more) problem(s). They were given a selection of possible problems appropriate for the age of their students but were also allowed to choose a problem on their own. The selected problems are displayed in the appendix of this article.

The teachers were asked to provide a rough lesson plan before observation started and they were interviewed after the observed lesson to explain differences in the planning of the lesson and its actual course.

At the end of the lesson, all materials used by the teacher (worksheets, overhead slides, etc.) or produced by the students (filled worksheets, problem solutions) were collected to be analyzed by the researchers.

The videos have been coded for activities (introduction, elaboration, classwork, group work, etc.) using the coding scheme of the Swiss-German video study (Hugener, 2006). In all plenary phases, the share of speaking time has been calculated (using a stopwatch). Additionally, the teachers' actions regarding problem solving have been coded using the grid presented above (see Table 1).

Two independent raters have conducted these codings (the classroom activities as well as assumed teachers' intentions) showing good to very good interrater agreement. If divergent codes appeared, the coders recoded the corresponding aspects of the lessons consensually.

To gather (2) data on the teachers' beliefs, the above-mentioned interviews contained questions regarding the teachers' understanding of the concept of „problems" as well as their views of mathematics: What is a problem? What is a good problem? What is problem solving? How important is problem solving for you? What does it mean to do mathematics? What should mathematics teaching look like? Additionally, the teachers are asked whether they were educated in solving problems on their own and in teaching problem solving. Finally, they were asked how often they use problems in their lessons. The interviews have been interpreted and discussed in small teams of researchers (*consensual validation*), especially the interpretations of Dionne (1984) about doing mathematics were important indices for coding beliefs.

Due to space reasons, only three teachers (named F, G, and H) and their lessons can be presented in detail (see the appendix for according codings). The codings of four additional teachers (A, C, D, and E, see Rott, 2016, for details) and their lessons are used to support the results obtained from those three lessons.

## RESULTS

Like previous chapters, the results chapter is divided into (1) a description of the lessons, focusing on the teachers' behavior, (2) analyses of the interviews, focusing on the teachers' beliefs, and (3) on the interplay of behavior and beliefs.

### (1) Description of the lessons

**Teacher F** allowed us to videotape two of her lessons of which both are addressed but only the second one will be presented in some detail, here. I chose the second lesson, because – in contrast to the first one – it involves non-routine problems. At the beginning of the first lesson, Teacher F announced: „In today's lesson, we are going to work differently than usual and solve some problems that – ahem – are more openly formulated than you are used to […]." This way, she reveals that she normally does not use problem solving in her lessons.

For both videotaped lessons, she gave work sheets with four to five problems to her students all of which could be solved by using linear equations. The topic of linear equations had been addressed in her teaching in the previous lessons but the teaching unit had not been finished. The problems of the second lesson can be seen in the appendix (Table 2).

In general, Teacher F did not interact much with the problem solving processes of her students who worked in groups. She did not help her students in understanding the problem or devising plans. She only highlighted that the students should be able to present their solutions at the end of the lessons.

In the lessons, she did not give content-related hints, but encouraged the students to pursue their ideas. In both lessons, trial-and-error approaches (drawings, tally sheets) were permitted (which was unfamiliar for the students). At the end of the lesson, Teacher F made sure that different groups with different approaches present their ideas. However, those approaches are not compared with each other.

Her actions resulted in the lesson coding as shown in Figure 1. The phases of Understanding, Planning, and Looking back are coded as „neutral", whereas the Carry out phase is coded as „emphasizing strategies" because of her encouragement of students' approaches. [The whole lesson lasted for 48 min.]

**Teacher G** started his lesson with a short announcement regarding the visitor (1 min) and then began to organize working on the problem for the lesson. He told his students that not speed is important, but presenting their approaches to their classmates. The students were allowed to decide whether they wanted to work alone or with up to three other students of their choice (2 min). In the time the students changed seats, the teacher distributed the worksheets for the lesson (2 min). The problem he chose for this lesson deals with four-digit palindromes and whether they are divisible by eleven (see Table 3 in the appendix). After the distribution, Teacher G explained for all students the term palindrome: the name „Anna" as well as the number „2552" are mentioned in this explanation (2 min). More information regarding the problem and possible approaches were not given by him. [This whole plenary discussion lasted for 7 min.]

In the subsequent working phase, the students worked on the palindromes problem. Most of the students chose to start with part b) that asks for the actual number of four-digit palindromes. During this phase, the teacher walked around and observed. When his students asked questions, he helped them in better understanding the problem (e.g., „The number 0-2-2-0 is no four-digit number."). However, he did not answer questions regarding the solution of the problem. At the end of this phase, Teacher G handed out overhead slides to groups that he chose to present their solution. [The group work lasted for 35 min.]

In the final phase of the lesson, the students were asked to present their approaches and their „first steps". Two students presented their solution; both students are very good as their grades show. After that, the teacher explained the approach of a third group (trying to divide all four-digit palindromes by eleven). After theses presentations, there were not many questions by the students and the teacher announced the problem as solved and the lesson as finished. [This plenary discussion lasted for 8 min.] [The whole lesson lasted for 50 min.]

The lesson was coded as shown in Figure 1: During Understanding and Carry out phases, the teacher encouraged a variety of different approaches. During planning and Looking back phases, he did not highlight strategies but results.

**Teacher H** started the lessons with an organization: The students should sit in a way that both single and pair work are possible (2 min). After that, he distributed the worksheet with the problem of this lesson (see Table 4) in the appendix. He also chose the palindromes problem but decided to add additional information for his students. He had a student read the problem formulation and then referred to „ANNA numbers" that the students should be familiar with from primary school. He asked his students to state some palindromes and made sure that palindromes with zeros were mentioned. He also explained that a four-digit number cannot start with a zero. After that, he explained what the students should show in a detailed way (7.5 min). Teacher H reminded his students how to prove or disprove mathematical statements: reasoning and proof or finding a counterexample (1 min). After the students had prepared their workplace (1.5 min), the teacher explained the following think-pair-share method: 5 min single work in silence, questions are only allowed in written form in this phase; after that, 15 min of pair work (1.5 min). [This plenary discussion lasted for 13 min.]

During the single work phase, the teacher only answered to questions in a very reserved way („you have to think about this again"). [Single work for about 9 min.]

Then, Teacher H announced „a short intermediate stop". He pointed out that the students have to reason their solution even though a substitute teacher once told them a divisibility rule for eleven [adding and subtracting the digits]. The teacher also announced that he had prepared aid cards. [Plenary discussion, 1 min.]

In the subsequent working phase, many students posed questions. Teacher H walked around and distributed aid cards. To some extent, he answered directly to questions, but always did so in a very careful way (e.g., „You have shown the divisibility for one number. The question is: is this is true for all [numbers]?"). Occasionally, he had to ask for silence. [Pair work, 18 min.]

After that phase, Teacher H gave a short summary: Some groups had tried to divide numbers by using the known algorithm and observed that the claim [every four-digit palindrome is divisible by 11] seemed to be true. However, it would not be the goal to calculate examples. It would be the goal to find reasons. With this suggestion, the students should start the group work. The teacher also reminded the students of the aid cards and gave them 1 min to rearrange the tables. [Plenary discussion, 4 min.]

During the following working phase, the teacher walked around and handed out aids taking care of not revealing a solution. He had to quiet the class several times. Three groups were given overhead slides to present their approaches. [Group work, 22 min.]

Then, Teacher H made the class stop working and talking and announced that in the presentations, the approaches were as important as the results. The first presenting group had only calculated 1001:91 = 11. The teacher explained why it is not sufficient to have only one example. The second group presented a complete solution: 1001, 2002, …, 9009 are divisible by 11, because you could always add 1001 (Teacher H reminded the students of the summation rule). The numbers in between can be reached by adding 110. The third group reasoned only with +110. The teacher pointed out that there is a gap in this reasoning. Since there were no further questions, the teacher asked if six-digit numbers could be look at with the same approach and had the students describe possible strategies to tackle this subsequent problem. [Plenary discussion, 14 min.]

In the time until the bell rings, the students should use their time to work on problem part b) (the number of four-digit palindromes). [Group work, 1 min.] [The whole lesson lasted about 80 min.]

In the understanding phase, Teacher H very closely managed his students and their understanding of the problem. In the interview, he told us that he chose to do so, because a lot of his students had a migration background and he knew their difficulties with reading problem formulations. In all other phases, he encouraged the use of different approaches and strategies without directly influencing the students' ideas (see Figure 1).

| Phase | closely managed | neutral | emphasizing strategies |
| --- | --- | --- | --- |
| Understand | Teacher H | Teacher F | Teacher G |
| Plan | | Teacher F G | Teacher H |
| Carry-out | | | Teacher F G H |
| Look back | | Teacher F G | Teacher H |

Figure 1: Profiles of the lessons by Teachers F, G, and H

### (2) Analyses of the interviews

For **Teacher F**, a problem is an open task, especially open regarding the possible approaches. For her, problem solving is similar to mathematical modelling; problems allow for individual approaches as well as individually correct solutions. However, she does not think that this idea of the term problem is generally accepted; therefore, she chose problems with only one correct solution for the filmed lessons. [Her problems do allow for different approaches, though.] Normally, her students only work in small groups, when they train known procedures and methods. She was surprised how well the students coped with working on unknown tasks. She was also surprised that even weak students were able to come to solutions. However, she would not use problem solving in classes dominated by low-achieving students. She observed that her students were not satisfied with solutions that were obtained by trial-and-error approaches; the students were expecting computational solutions. She blames the school system that, generally, non-computational solutions are not appreciated. However, she adds that in written tests, she would not accept trial-and-error solutions, either. She also expresses the belief that especially low-achieving students would need this kind of mathematization and structure [i.e., algorithms and computation schemes]. This interview was hard to rate in respect to the beliefs of Teacher F, because she does not talk much about mathematics. The two coders consensually coded her beliefs as holding an *instrumentalist view* because of the way she talks about problems and the way she highlights calculatory solutions to problems and the use of formulas which is in accordance with Dionne's (1984) interpretation of this belief (see above).

For **Teacher G**, a problem is „a task that does not really fit into the teaching contents and that can be solved with tools and ideas that have been obtained some time ago. […] A task for which one has to see what tools from mathematics are needed [to solve it]." This idea of the term „problem" fits to the other beliefs that Teacher G expressed regarding the learning of mathematics: teaching is successful, „when students have internalized special techniques so that they can use them." It is important for him to design his teaching in a way that mathematics is linked to everyday life so that students always know what procedures they should use. Generally, the interview revealed an *instrumentalist view* of mathematics. For the teaching of Teacher G, this belief manifests in the following way: When problems are used, they are used at the beginning of teaching units. The tools and procedures to solve such a problem are then developed within the unit. The students learn this way „[which tools] are needed to solve it [the problem from the beginning] well and fast." This fits the interpretation of Dionne (1984), doing mathematics as using rules and procedures. For Teacher G, problems that cannot be solved with the help of techniques from a previous school year or that are not intended to develop and learn new techniques cannot be used in regular teaching. Maybe, such problems can be used in special working groups for enthusiastic students that want to participate at mathematical Olympics. But those students would know such problems already.

For **Teacher H**, a problem is „a kind of mathematical puzzle." He adds immediately that „finding out something new" belongs to the activity of problem solving. He does not reflect about his own activities – as far as he remembers, he never learned about problem solving in his education – but only about problem solving in mathematics lesson. For him, problem solving is synonymous with „developing a strategy to successfully develop a solution". As examples for strategies, he mentions working forwards and backwards. Acquiring competencies in such a way enables students to solve „problems that build on previous problems or problems that are more complex." Fittingly, Teacher H thinks that problems are good problems, if they provide the opportunity „to develop strategies" or when they help demonstrate „exemplary general mathematical strategies or ideas." Finally, problem solving is important because it provides the opportunity to „network and further develop knowledge," because only via problem solving one can „overcome a receptive [algorithmic] approach." Using the characteristics of the perceptions by Dionne (1984), Teacher H thinks of doing mathematics as developing thinking processes and finding relations between different notions. Overall, the beliefs of Teacher H regarding mathematics and learning of mathematics, respectively, can best be described with the *problem-solving view* by Ernest. This also fits his teaching style, because he designs his teaching in an open way and does not perform planning and implementation phases for his students – at least in the two lessons (and especially in the second lesson that has been described above) that have been observed. For example, he predicted different difficulties of his students (e.g., linguistic understanding of the task formulation, formalization of the own approach, and usage of required prerequisite knowledge) and prepared corresponding aids. However, he did not give those aids and hints „to all students from the beginning [of the lesson]"; instead, he prepared those aids for being able to help individually. In the interview, Teacher H said that he uses two to three teaching units with an emphasis on problem solving per school year.

**(3) Interplay of teachers' behavior and beliefs**

The behavior of all three teachers presented above fits their beliefs. For example, Teacher H – holding the problem-solving view – shows actions that emphasize strategical diversity and different approaches to solve the problem by the students, especially during a final „look back" phase. Both Teacher F and G have been identified as holding the instrumentalist view of mathematics and both teachers do not focus on strategies and approaches at the end of the lesson but do highlight the results.

In a more general way, the data of all seven teachers presented in Figure 2 show a remarkable resemblance between their behavior and their beliefs.

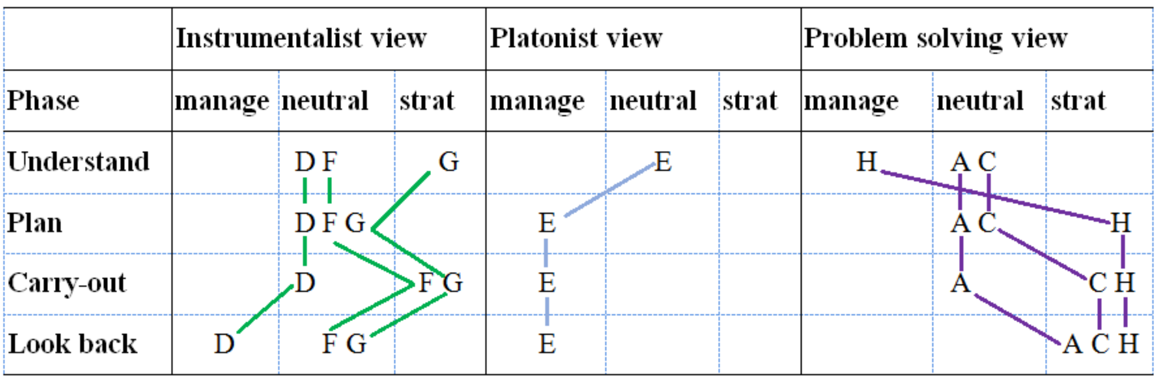


Figure 2: Profiles of the lessons by Teachers sorted by their beliefs

Teachers holding the instrumentalist view (D, F, and G) seem to care more about results than about approaches. In the look back phase, after a problem has been solved, they do not highlight strategies. It is important that the students know how to solve a problem. Often, they seem to choose students with correct approaches to present their results to the class. In addition, if – in the course of this study – they use a problem chosen by themselves (not selecting one from the list we provide), this problem seems to be related to important content (e.g., the Pythagorean Theorem or solving linear equations).

Teachers holding the Platonist view seem to think that there is one correct solution and one correct way for students to solve a problem. That approach has to be found by the students and the teacher makes sure the students find that way by giving concrete hints to help them (e.g., by making excessive use of hint cards).

Teachers holding the problem-solving view seem to care more about approaches than about results. They do not highlight results after a problem has been solved. In contrast, strategy use and different approaches are emphasized in the looking back phase. Often, they seem to choose students with partial or incorrect approaches to present their results to the class.

## DISCUSSION

In this article, by using the grid (see Table 1), I introduced a new way of analyzing teacher behavior in lessons with the subject problem solving. In previous publications (e.g., Rott, 2016) other kinds of measure like the sequence of phases in the lessons (plenary time, single work, group work, etc.) and the share of speaking time in plenary phases have been used to characterize the behavior of the teachers in an objective way. However, these measures did not produce meaningful results in pointing out the interplay of teachers' behavior and their beliefs. Using the grid shows a conclusive correlation (in a non-statistical sense) between teachers' actions and their uttered beliefs (see Figure 2). The grid could be an important contribution to classify types of teaching regarding problem solving.

So far, the results rely on only seven teachers and their lessons, which is a clear limitation of the study presented here. Nonetheless, these results are valid insofar as they fit the predictions by Ernest regarding the interplay of mathematical views and behavior of teachers (see section (3) in the theoretical background). More lessons have already been recorded and will be analysed soon.

Additionally, this study has been replicated with Indonesian teachers, also revealing a significant interaction between teachers' beliefs and their behaviour in lessons dedicated to mathematical problem solving (see Safrudiannur & Rott, 2017).

### *Additional Notes*

I thank Safrudiannur for proofreading, for very helpful recommendations, and for helping me finding the quotations by Paul Ernest (in the theoretical background). The lesson and beliefs of Teacher B (not included in Figure 2) are discussed in Rott (2015).

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## APPENDIX

### Teacher F

Teacher F is a teacher of about 40 years with about 10 years of teaching experience that teaches at a bilingual Gymnasium in Neuss. She chose a 8th grade class for her lesson and used problems that involve linear equations (see Pohl, 2015, for details). [Two lessons were filmed, the second involved problem solving.]

Table 2a: The chosen problem; teacher F

| **Reindeers and Humans** | **Candles** |
|---|---|
| a) In a winterly story, there is a feast with people and reindeers. Overall, there are 22 heads and 60 legs. How many humans and how many reindeers attended the feast? | b) Marlon lights a candle of 33cm that burns off 5.5cm per hour. What height does the candle have after two hours? How long until it is completely burned down?<br>c) Marlon's sister lights another candle at the same time. After one hour, it has a height of 23cm. After three hours, its height is 15cm. What was this candle's height at the beginning and how much cm does it burn off per hour?<br>d) When do both candles have the same height? |

Table 2b: The planning and the actual course of the lesson; teacher F

| **Planning** | **Actual Course of the Lesson** |
|---|---|
| Introduction | Introduction, announcement of the filming. Introductory task, checking the homework *(10 min)* |
| Organization: Information regarding the course of the lesson | Information regarding the lesson: „Same procedure as in the previous lesson. The way you approach the problems is not important. However, you should write your solutions on a slide and be able to present them." *(1 min)* |
| Working on the problems: group work using the worksheet „a cold winter's day"; allocated time 25 min | Group Work – students work on the worksheet *(25 min)* |
| Presentation: correction and comparison of the results | Presentation of different approaches on overhead projector slides by different groups *(12 min)* |
| Total: *45 min* | Total: *48 min* |

Table 2c: Cumulative duration and share of speaking time; teacher F

| **Activity** | **Cumulative duration** | **Share of speaking time** | |
|---|---|---|---|
| Classwork | 12:51 (33.8%) | Teacher: | 02:52 (23%) |
| | | Students: | 05:03 (39%) |
| | | Silence: | 04:56 (38%) |
| Single work | --- | | |
| Pair work | --- | | |
| Group work | 25:09 (66.2%) | | |
| Total | 38:00 (100%) | *[counting the ps part, only]* | |

## Teacher G

Teacher G is a teacher with more than 13 years of experience teaching at upper secondary school (Gymnasium) in Mülheim. He has no experience with problem solving from his studies or professional teacher development courses. The lesson was filmed in his grade 7 class that he has been teaching for two years (cf. Kretz, 2016).

Table 3a: The chosen problem; teacher G

| **Palindromes** |
|---|
| a) A palindrome is a number that reads the same backwards as forwards. A typical example is 2552. A friend of mine claims that all palindromes with four digits are exactly divisible by 11. Is he right?<br>b) How many palindromes with four digits exist?<br>[Source: Mason, Burton & Stacey (2010, S. 4 f.) – slightly varied] |

Table 3b: The planning and the actual course of the lesson; teacher G

| Planning | Actual Course of the Lesson |
|---|---|
| 1. Entry: presentation of the problem using an overhead projector. Goal: Not the solution but esp. the approach to solve the problem is important.<br>Plenary discussion | Organization of the filming *(1 min)*<br>Organization of working on the task, introduction of the problem – classroom discussion *(6 min)* |
| 2. Elaboration: solving the problem, preparation of overhead transparencies<br>Worksheet, self-chosen social form | Elaboration – single, partner or group work *(35 min)* |
| 3. Securing the results: students present their approaches comprehensibly.<br>Plenary discussion, gallery walk | Securing the results – different groups present their approaches – plenary discussion *(8 min)* |
| Total: *45 min* | Total: *50 min* |

Table 3c: Cumulative duration and share of speaking time; teacher G

| Activity | Cumulative duration | Share of speaking time | |
|---|---|---|---|
| Classwork | 15:00 (30.0%) | Teacher: | 09:18 (62%) |
| | | Students: | 02:50 (19%) |
| | | Silence: | 02:52 (19%) |
| Single work | --- | | |
| Pair work | --- | | |
| Group work | 34:59 (70.0%) | | |
| Total | 49:59 (100%) | | |

## Teacher H

Teacher H is a teacher with more than 10 years of experience, at his school in Essen he is also a coordinator. He has no experience with problem solving from his studies or professional teacher development courses, except for maybe courses using a graphic calculator. The lesson was filmed in his grade 6 class.

Table 4a: The chosen problem; teacher H

| **Palindromes** |
|---|
| A palindrome is a number that reads the same backwards as forwards. Maybe you know those numbers with four digits from primary school as ANNA numbers<br>A typical example is 2552, exactly like the letters in ANNA. […]<br>a) Mr. Breuer [a colleague] recently claimed that all palindromes with four digits are exactly divisible by 11. With your help I want to show: Is he right?<br>b) Mr. Breuer also claimed that there are 100 palindromes with four digits. Is he right?<br>[Source: Mason, Burton & Stacey (2010, S. 4 f.) – varied by the teacher] |

Table 4b: The planning and the actual course of the lesson; teacher H

| **Planning** | **Actual Course of the Lesson** |
|---|---|
| Goal: The students use the addition rule to give a reason the divisibility of four digit palindromes by 11. | Organization: Enable single and group work; distribution of the worksheet. |
| Ensure understanding *(10 min)* | The teacher explains the problem; he reminds the students of mathematical reasoning – plenary discussion *(13 min)* |
| Working on the problem, Think-Pair-Share method *(45 min)* | Single work *(9 min)* / plenary discussion *(1 min)*<br>Pair work *(18 min)* / plenary discussion *(4 min)*<br>Group work *(22 min)* |
| Securing the results, comparison *(15 min)* | Securing the results, Comparison – plenary discussion *(14 min)* |
| Problem part b) (group work) *(10 min)* | Problem part b) – group work *(1 min)* |
| Total: *80 min* | Total: *80 min* |

Table 4c: Cumulative duration and share of speaking time; teacher H

| **Activity** | **Cumulative duration** | **Share of speaking time** | |
|---|---|---|---|
| Classwork | 31:25 (39.4%) | Teacher: | 17:55 (54%) |
| | | Students: | 08:04 (25%) |
| | | Silence: | 06:21 (21%) |
| Single work | 08:42 (10.9%) | | |
| Pair work | 17:51 (22.2%) | | |
| Group work | 21:57 (27.5%) | | |
| Total | 79:55 (100%) | | |

# FORMATION OF THE FUNCTION CONCEPT THROUGH CONCRETE PROBLEM SITUATIONS

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*The concept of function is a fundamental base in the process of learning mathematics, which can not be acquired in one step. Formation of it is a long process.*

*Studies are done of function concept formation processes in Ukrainian and Hungarian curriculum and textbooks showed that the Hungarian curriculum and textbooks prescribe more targeted preparatory work from the elementary school to establish the concept of function. The investigations among one class of Hungarian and one class of Ukrainian students in 6th and 7th grades concluded the source of the concept incorporation into the teaching process strengthens the formation of the correct function schema. According to the results of our studies we prepared and implemented an improvement program on the preparation of the function concept among 6th grade students from Ukraine. Within the program, we aimed the concept formation with the help of concrete problem situations. At the same time we developed the skills necessary for problem solving: communication skills (development of the relational vocabulary, reading comprehension and interpretation of the text).*

*In our paper we are examining the effect of the improvement program on the formation of the function concept and on the development of the skills, mentioned above after 1 year (in 7th grade).*

## INRODUCTION

While teaching Mathematics, we have frequently faced the problem that students before graduation from elementary or secondary schools and even first year university students are lost in the world of functions. This experience is confirmed by several researches by which students face many difficulties in understanding function at different ages of secondary education (Sajka, 2003, as cited in Panaroura et al., 2015) and they can hardly or even cannot apply the concept in solving problems related to functions (Panaroura et al., 2015, Elia et al., 2007) According to Nachlieli and Tabach (2012), to introduce the function concept an informal basis must be established upon which a new mathematical topic can be built later. Thus, preparatory work is necessary for the introduction of this concept, and this work must start in the years preceding the creation of a definition.

While analysing the Ukrainian and Hungarian mathematical curriculum and textbooks which were used by students who participated in our 3 years longitudinal research, it turned out that this preparatory work is less stressed in Ukrainian curriculum and textbooks than in Hungarian ones (Szanyi 2015, 2016). Based on our experiences it becomes clear that the preparatory work of the examined Hungarian group of students (which has started from elementary school) had a positive effect on the formation of the correct function schema (Szanyi 2015, 2016). They are capable to recognize the relationship between covariant quantities in several problems related to functions, to switch flexibly between several representations (from iconic to symbolic and conversely).

By considering such experiences we have worked out an improvement program which aimed the preparatory work for the formation of function concept. The program was executed in a 6$^{th}$ grade of a Hungarian school of Ukraine in school year before defining the concept itself.

With the help of a knowledge and skill-based examination, we were looking for the answer of the following question:

*Did the improvement program influence*

*(a) the formation of the function concept after 1 year the execution of the program;*

*(b) the skills which are necessary for the problem solving, namely communication skills (development of the relational vocabulary, reading comprehension and interpretation of the text), flexible switching between representations, recognition of relationships.*

## THEORETICAL FRAMEWORK

Mathematical definitions play an important role in the study of practically every area of mathematics (Usiskin & Griffin, 2008; Vinner, 1991). Many researches emphasize (Skemp, 1971; Sierpinska, 1992) that an introduction of a definition has no meaning up until the student has the significant level of mathematical culture. According to Skemp (1971) students must learn the meaning and significance of the words in the definition before introduction of the concept and its relation to ordinary words. All of this becomes available with the help of specific, unique examples. The role of examples is significant in the process of abstraction since comparing them and emphasizing their common properties help reach the concept inductively. According to Tamás Varga (1969), it is better to start this process with specific, concrete things, so on the enactive level (Bruner, 1968). Without proper preparation, later symbols will mean empty formulas and the contextual meaning of definitions may be lost.

It is no different in the case of function concept. Children don't use the word 'function' in their stages of development, so they don't have a preconception of it either (Nachlieli & Tabach, 2012). Thus, proper preparatory work is necessary for its introduction and its the best to start it years before defining the concept in a spiral way.

The didactic analysis of the function concept pointed out how the mathematical culture can be build up to introduce the definition. By examining its historical development (Kleiner, 1989; Sain, 1986) constitutive parts of the definition stand out which contain prior knowledge necessary for comprehension (Sierpinska, 1992). Most concept formation processes in the school follow their historical development. It is the case with the function concept too. Kleiner (1989) uses three mental pictures to describe the historical development of function:

- geometric (Leibniz in 1692 used the function to denote any quantity connected with a curve).
- algebraic (Bernulli and Euler defined the function as an analytic expression, which is a formula describing the relationship between constants and variables).
- logic (Dirichlet and before him Farkas Bolyai developed the function concept used today).

So, we use the „narrow” function concept (see Bernulli and Euler definition) to prepare the introduction of function at school, that is, relations that can be described by rules are discussed first. The „broader” function concept (see Dirichlet definition) emerges from this (it is not important anymore how the relation evolved).

According to Sierpinska’s statement, students must familiarize with the following „worlds” which contain the constitutive parts of the function concept:

- World of changes or changing objects – In the definition of the function the two variables (dependent and independent) represent the changing objects.
- World of relationships or processes – The relationship between changing objects or the process which transform objects into other objects can be described for example by table, graph, formula or can be expressed in verbal form.
- World of rules, patterns, and laws – The relationship must be well defined by a rule which specifies how the first or independent variable affects the second or dependent variable.

The detection of change and the recognition of the relations between variables is important in the concept formation process. The „quality” of change can be determined by the rules describing the relationship between the variables. Sierpinska writes that „…the most fundamental conception of function is that of a relationship between variable magnitudes…” (as cited in Sierpinska, 1988, p. 572).

The difference between the rule and relationship is subtle because it is possible for one to detect a relationship but fail to explicity state the rule. On the one hand, finding rules, patterns and laws can be used as an entry point to the development of the function concept (Kwari, 2007). On the other hand, the skills connectable to these parts (recognition of connections, recognition of rules (searching patterns), communication) are essential for problem solving, too.

There are different definitions for the term „problem solving”. Most of them include a starting point but the way of reaching the goal is hidden for us (Pólya, 1985, Johnson, 1972). There are a number of models for the process of problem solving in the literature (Schonfeld, 1985, Pólya, 1945, Mason, 1961). In the models, the process of problem solving is determined by sets of several steps. These steps include – among others – creating a figure (any iconic representation) or interpreting a given figure, searching patterns as heuristic strategies („methods and rules of discovery and invention") (Pólya, 1945, p. 112).

Applying these heuristic strategies can be aimed also in the function concept formation process when introducing its parts.

The function concept has so called cognitive roots which can help to understand the constitutive parts mentioned above, and they play a role in developing the necessary skills to comprehend the concept on a deeper level (such as switching between different representations, rule recognition (finding patterns) and rule following). Tall (1992, p. 497) defined a cognitive root to be „an anchoring concept which the learner finds easy to comprehend, yet forms a basis on which a theory may be built." This is a meaningful cognitive unit of core knowledge for the student at the beginning of the learning sequence (as cited in Tall, McGowen & DeMarois, 2000, p. 257). One of the cognitive roots of the function concept is the function machine which objectifies the definition and helps in familiarizing its constitutive parts. Even (1998) claims that the skill of identification of one concept and its representation in different forms as well as the flexibility from one representation to another makes it possible to understand the concept on a deeper level (as cited in Elia, 2007, p. 538). The function machine can be a tool to achieve this goal since it makes all the other representation of the function available (verbal, table, arrow diagram, formula, graph).

Another cognitive root of the function is a process taken from real life. Some researchers (Davidenko, 1997; Van de Walle, 2004) confirm that discourses on real-world phenomena involving covariant quantities should be promoted in the concept preparation process, in conjunction with the discourses symbolic expressions and on tables.

Based on the literature review about the formation of the function concept we can claim the following:

- The constitutive parts of the concept should be placed in the focus of the preparation phase.
- The introduction of the constitutive parts and the reification of the the concept of covariant quantities can be carried out by the involvement of the sources of the concept (function machine, process taken from real life).
- The different representation of the function (verbal, table, formula, arrow diagram, graph) can facilitate the recognition of the relationship between covariant quantities.

- We also must keep in mind the improvement of the rule following and rule recognition skills which can be connected to the rule which describes the recognised relationship between the quantities.
- The skill of switching between different representations should also be improved, which help the deeper understanding of the concept and it has an important role in problem solving, too.

In conclusion the formation process of the function concept needs careful work. These statements were based on the planning of activities (which will be presented later) to be used in the program for preparing the function concept by building certain characteristics of the problem-based learning (Molnár, 2005) in the program.

## THE BACKGROUND OF THE RESEARCH

In the Ukrainian and Hungarian (hereafter referred to as UA and HU) education system, function as a mathematical concept is defined at the seventh grade of the secondary school. In the lower classes, students are prepared with the use of different materials for the introduction of the function concept. According to the above-mentioned studies, we analysed those HU and UA curricula and the textbooks for the fifth and sixth grade from the viewpoint of formation of function concept, which were used by our research participant students during their studies. Based on the tests it can be stated that both countries' curriculum and textbooks have materials that help in the preparation of the concept. However, this amount is smaller in Ukraine compared to Hungary. The improvement of the skills switching between different representations, rule recognition and rule following skills which are necessary for the preparation of our concept are less stressed (in lower classes completely absent). So, the concept formation happens in a targeted form in the case of students from Hungary, while in the case of students from Ukraine it's different.

Benő Csapó (Falus, 2004, 292. old.) divides the students' knowledge examined with tests into two main spheres. One of them includes the knowledge-based components (images, definitions, concepts, etc.) and the other one includes skill-based components. The latter group contains different skills, abilities, proficiencies. Based on this and within the framework of comparitive pedagogical research with longitudinal research, we have examined a Ukrainian student group's in a school with Hungarian as the language of instruction and Hungarian student groups' development of functional concept for three years (from 6th to 8th grades). The chosen student groups identically studied mathematics four times a week. Examinations in the sixth and seventh grades aimed to measure the knowledge of constitutive parts of function concept and connected skills related to the recognition and following of rules, i.e. the development of functional schema (Szanyi, 2015, 2016). This was followed by the examination of the 8th grade students' concept image about function (Szanyi, 2017). The tests – which both student groups participated in – used in these examinations were a mixture of the two knowledge types mentioned above. On the one hand, we focused on their knowledge of the function concept. They have aimed to measure their knowledge and familiarization about the constitutive parts of the function concept as well as the students' concept image about the function itself. On the other hand, with these tests we have also examined skill-based knowledge which is characterized by being on a steady-state after a long development stage from the beginning. When we measure abilities and skills we can only state some students' current stage in this development (Falus, 2004, p. 293.). Besides measuring the development of their knowledge, the tools used within the comparative pedagogical study also focused on the development of the following skills:

- communication skills (development of the relational vocabulary, reading comprehension and interpretation of the text)
- skill of different representation of one specific problem
- coherence recognition skill (pattern recognition skill)

Results have shown that due to the insufficient development requirements the investigated 6th grade UA students possessed the rule recognition and the rule following skills on a lower level than the investigated HU students. The UA students were able to recognize and to follow the relationship between the covariant quantities which were represented in different ways, but the argumentation verbally, which requires the communication skills or with formulas for the rule that defines this relationship were limited and caused some problems for them. The function, which was introduced in 7th grade, could be built on stronger fundaments in the HU group. The introduction of the function concept in 7th grade had an impact on the recognition of the relationship between the covariant quantities represented in different ways among the UA students. But the performance of the UA students was behind the HU students in 7th grade also. Thus, the targeted preparation work strengthened the formation of the right function schema in the HU group. But according to the results of the investigation of the 8th grade, despite of the differing concept formation process the HU students couldn't form a more ideal concept image of function by the end of the 8th grade than the UA participants (Szanyi, 2017).

## METHODOLOGY

### Previous work – an improvement program

Considering the results mentioned above, we have planned an improvement program designed in an action research, targeted on the preparation of the function concept in another 6th grade group (11-12 years old) of Hungarian school of UA (Szanyi, 2017). Since the goal of the action research is to solve a special problem in a given environment with its result being useful in other practices (Falus, 2004), a „real" control group was not included in the research. The goal of the program was to be put to a trial in a given environment and – when considering the experiences – to actualize it in several student groups.

In the lessons, the children worked on their own or they were guided by a teacher. Individual work was monitored constantly: reports were made about each lesson, furthermore, occasionally voice records and photos were taken. The important „moments" were put down in black and white, too. In the first 6 lessons, children made their work into a notebook („Fun") introduced specifically for the improvement program. We made several worksheets for the lessons which were allocated for each student. These were stuck into the „Fun" notebooks in advance or the students did it during the lessons.

When organizing the lessons, we adjusted to certain characteristics of the problem-based learning. Problem-based learning has a number of definitions. Non-exhaustive, one of them is mentioned here: „environment of learning where the problem is the power of learning" (Molnár, 2005, p.32)

- the work of the students are guided by the tutor (teacher);
- in the lessons the students were given problems of real life which are built on each other (Szanyi, 2017) based on the principle of spirality (Ambrus, 2004). For example, on the fifth occasion the following problem was used:

> Task: Peti was talking on the phone with one of his friends in a 5-minute break. The duration of their conversation is measured in whole minutes. 1 minute costs 0,5 hryvnia. How much could the conversation cost to Peti, if the connection fee was 1 hryvnia? 1) Arrange covariant quantities (element matches) in a table. 2) Express it a) with words, b) using the language of mathematics (with a mathematical expression), how much Peti had to pay if the duration of the conversation was x minutes?

- the problems were adapted to the recognition of the parts of the concept and to the development of the problem-solving skills.

After each lesson, we collected the notebooks so we could analyse the works of the students together with the voice records and the notes, every one of them. This way, we could monitor the improvement of the children constantly.

In the Table 1 we summarized the improvement program of 11 occasions.

| | **Improvement strategy, method** | **Brief content description** |
|---|---|---|
| **An exploratory study (17 November 2015)** | | |
| **1st part of the improvement program (23 November – 7 December 2015)**<br>**Topic in the curriculum: Ratio, ratio pair**<br>(the sessions covered only part of the lessons) | | |
| 1. | Analysing skills, relational vocabulary improvement | Creating element pairs with manual activities |
| 2. | Formation of the concept of covariant quantities, improvement of relationship recognition skills and recording them in different ways | The introduction of covariant quantities with the use of function machine |

| | | |
|---|---|---|
| 3. | Improvement of the rule following skill, and of the switching between different representation forms | Defining the substitution value of expressions |
| 4-5. | Improvement of the recognition of the covariant quantities and the relationship between them | Recognising covariant quantities hidden in textual tasks and representing them with tables, formulas |
| 6. | Improvement of „two-directional” thinking | Connecting situations and algebraic expressions which describe their solutions |

| **Post-test (19 January 2016)** | | |
|---|---|---|
| **2nd part of the improvement program (25 April – 4 May 2016)**<br>**Topic in the curriculum: Graphs** (sessions covered the whole lesson) | | |
| 7. | Improvement of observation skills, understanding everyday graphs | Introducing graphs. Reading data from graphs |
| 8. | Recording covariant quantities. Improvement of the switching skill between representations | Creating a graph based on a prepared table and filling in a table based on a graph |
| 9. | Improvement of the recognition skill with playful methods | Acting out situations to prepare the table and graph representation of covariant quantities |
| 10. | Creating experimental functions, representing them and connecting them with different situations | Representing different situations taken from real life (discrete case) |
| 11. | Creating experimental functions, representing them and connecting them with different situations | Representing different situations taken from real life (continuous case) |
| **Post-test (11 May 2016)** | | |
| **Delayed test (23 May 2016)** | | |

Table 1: The improvement program designed to prepare the function concept

According to the experiences of the improvement program based on the observation of students' work in the classes, the post-tests and the delayed test we could say, that the participants of the program at the end of the 6th grade reached similar results to the HU students who were the participants of the comparative pedagogical research. Also, it should be mentioned, that after the improvement program for the purpose of preparing of the function concept according to the textbook and curricula did not have any targeted work among the participants of the program.

**Procedure**

To get an answer to the question at the end of 7th grade, the participants of the improvement program completed the same test which was used among the 7th grade students of the 3 years long longitudinal research (Szanyi, 2016).

The research was carried out at the end of the academic year of the seventh grade, in May (on the last mathematics' lesson of the academic year). A written test was used which contained six tasks and students had 40 minutes to complete that. The test was completed by 17 students. The base of the tasks were recognition or expression function-like relationships between covariant quantities, represented in different ways; without mentioning the word „function" in the instructions (but the concept of function had been introduced). In order for us to be able to examine the effects of the program on the development of the function concept, among the tasks we will analyse those which are satisfies the following conditions:

- similar to the tasks that were used during the improvement program
- were used in the delayed-test of this program
- were used in the comperative pedagogical research among the 6th and 7th grades HU and UA students.

*Task 1. Find a relationship between the x and y values of the columns and based on it, complete the table with the missing elements.*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *x* | *16* | *12* | *4* | *-8* | *20* | *4* | *0* | | | |
| *y* | *4* | *3* | *1* | | | | | *6* | *-3* | *-1* |

*Write down the relationship: a) with words; b) with formula! Plot the pairs of values on a coordinate plane!*

This task targeted the recognition, application, representation of a relationship between covariant quantities in verbal form, with formula and graph. That is, the task requires the knowledge of all three constitutive parts of the function concept: variables, relationships and rules. The filling in of the blank places of the tables assessed the recognition of the relationship and the application of a rule (e.g. determining *y* values based on *x* values and vice versa), that determine of this relationship. Furthermore, it provided us with the opportunity to study whether students were able to switch between the representation forms (such as table, verbal, formula and graph). It's true that some of the coherent parts don't determine a function clearly, it is possible that more than one rule is made that defines the relation between the values. The goal was that the students had to find a proper rule which defines the relation of variable values (*x* and *y*).

*Task 2. Find a relationship between the x and y values of the columns and based on it, complete the table with the missing elements.*

| *x* | *1* | *10* | *7* | *0* | *9* | *20* | *38* |
|---|---|---|---|---|---|---|---|
| *y* | *5* | *23* | *17* | | | | |

*y*=__________________

*Write down the relationship: a) with words; b) with formula.*

Similarly to Task 1., this task required students to find a relation of covariant quantities represented in the table, but the relation had to be made with two steps of rules. Furthermore, the task also urged students to switch from one representational mode to another (from table representation to verbal and formula representation mode) that is the skill of different representation of one specific problem. The way students could write down the relation was a good way to measure their communication skills.

*Task 3. Fill in the table according to the following rule:* $y = \frac{x}{3} - 7$*. Write down the rule in words!*

| *x* | *9* | *0* | *42* | *63* | *15* | *3* |
|---|---|---|---|---|---|---|
| *y* | | | | | | |

The task aimed to examine the recognition of the relation of covariant quantities, the rule following and the communication skills.

*Task 4. 2 litres/second of water flows from a tap to a tank. How much water is in the tank at:*

*a) 1 s,* *c) 5 s,* *e) 16 s,*

*b) 2 s,* *d) 10 s,* *f) x s*

*later if the tank was empty at the beginning? Illustrate the relationship between the amounts*

*I) in a table.*

*II) with a formula.*

*III) How much water is in the tank when, at the moment of opening the tap, the tank contains 3 litres of water? Use a formula to describe the relationship between the values.*

In this task, covariant quantities are hidden in text. The student recognizes the covariant quantities if he/she represents them with table (the I) question of the task). The correct answer given to II) refers to the student's ability to generalize the task which means that he/she can represent one specific problem differently. Question III) focuses on the recognition of the multiple-step rules that define the relation of the covariant quantities and its transformation into a formula. So, in this exercise the students had to find a rule (pattern) defining the background of the problem.

Students had already faced similar problems detailed above in 3., 4. and 5. occasions in the improvement program. Considering their type, the same tasks were given in the delayed test that focused on the effectiveness of the improvement program in sixth grade.

## RESULTS

The Figure 1 represents the success of the tasks in which the recognition of the relationship of covariant quantities and following the rule which define the relationship were requested. Furthermore, the diagram compares the rate of right answers given by UA and HU student group participants of the comparative pedagogical research (results of 2014-2015) and compares the rate of correct answers of UA students who participated in the improvement program (result of 2016-2017).

As we have already mentioned before, the HU students who participated in the longitudinal research, the specifical formation of the function concept began from the first grade of the primary school. In the case of UA students, this preparatory work was less emphasized.

On the following figures, the results emphasized by black frames show the results of the students of the program in the 6th grade, right after the program and in the 7th grade, 1 year later.

Based on Figure 1 it can be stated that the 6th grade UA students that participated in the 2014 measurement were less successful compared to the 6th grade UA students who participated in the improvement program when it recognized the relation of the covariant quantities in the table or in context.

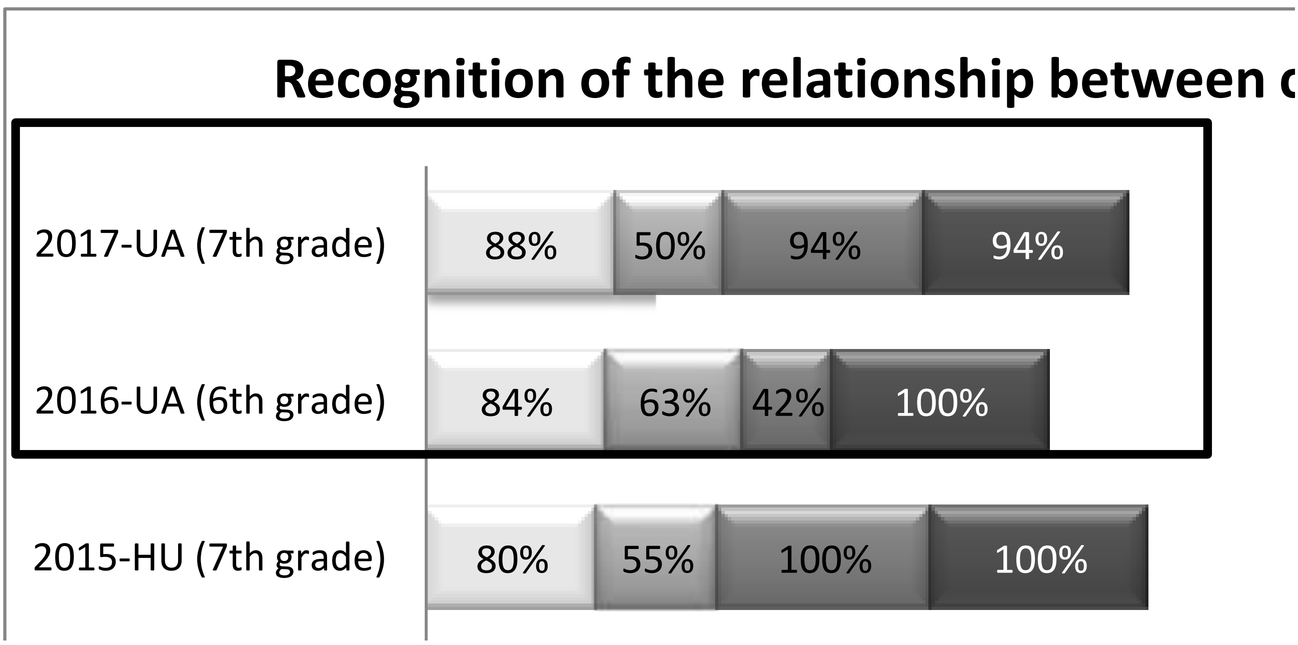


Figure 1

The results of UA students in 2015 show the effects of the introduction of the function concept along with the spontaneous development. While in 2014 the comprehension and following of explicite rule posed a problem to many UA students, in 2015 after the introduction of the concept they successfully answered the related question (Task 3). We have experienced the same with improvement program participants: the development of comprehension and following of explicite rule was affected by the introduction of the function concept.

It must be emphasized that the recognition of the relations' constitutive parts was served well by the activities presented in the improvement program since the results of UA students in 2017 were already closer to the 7th grade HU students (measured before) and their results clearly show the effect of specified preparatory work in the recognition of relationship of constitutive parts.

Although we can see, while in 2014 it was only 8% of the students who recognized the multi-step rule in 2017 already the students got closer to the 2015 results of the HU students, still the two-step rule makes it possible to define the slight downturn in the recognition of the relation among program participant students. This task represented serious difficulties particularly to those 7th grade UA students who did not recognize the relationships of covariant quantities in the delayed test which focused on the effectiveness of the improvement program. Because of that we believe that an expansion of the improvement program's courses is necessary to develop the recognition of relationships defined by the two-step rule.

Figure 2 shows the solvability of tasks that requires the recognized relation's verbal representation, that is, the change in students' communication skills. Because the verbalization of rules provides us with the opportunity to facilitate „transitioning from a verbal expression to an algebraic rule" (as cited in Rivera, 2013, p. 76), and that's why the development of communication skills of students in the process of the formation of function concept is highly essential. As Figure 2 shows, 6th and 7th grade students who participated in the improvement program verbalize the recognized rule in higher quantities compared to those UA students to whom the preparatory work of the concept was not targeted. But within a year no significant development was shown in the skills of UA students who participated in the comparative pedagogical research, and the same applies to those who participated in the improvement program. What's more, we have experienced a slight downturn. The reason for this is the greater emphasis for the algebraic terms (identities of power, transformation of expressions, etc.) in the mathematical curriculum of the 7th grade.

However, if the algrebraic concepts don't get enough attention it may have a bad result in the development of students' communication skills.

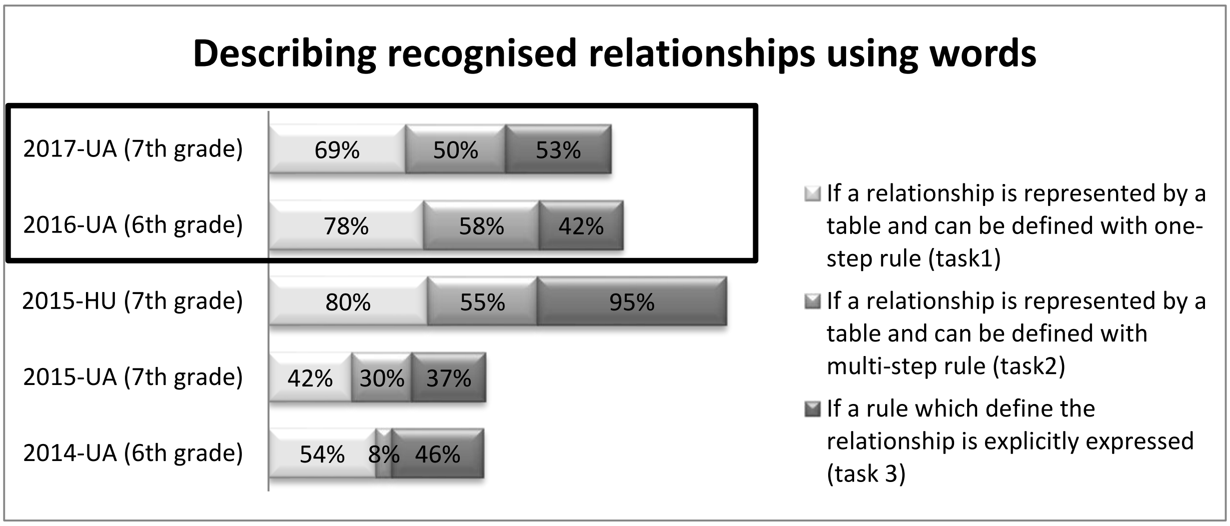


Figure 2

For example, the type of the tasks as Task 3 are quite common in the textbooks after introduction of concept, but there is no requirement for the description of the rule with words. So, the students are also disregarded from the argumentation verbally in this case. However, a great number of students could understand the explicite rule (see Figure 1), but they have avoided defining it or they have identified it with symbols instead of describing it with words (Figure 3) (definition of BB: „$y = \frac{x}{3} - 7$ based on this I have done the calculations”).

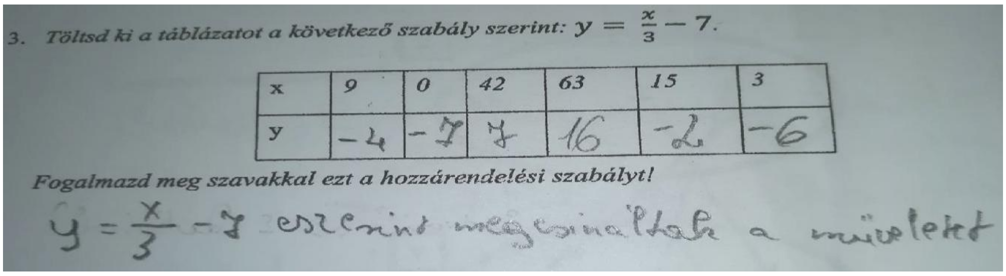


Figure 3

In the description of the recognized rule using formulas the effect of the program could have been noticed (see Figure 4). While the symbolic description of the rule defining the recognized relation was problematic for 6th and 7th grade UA students who participated in the longitudinal research, participants of the improvement program – if they recognized the relation between covariant quantities – had a higher rate of presenting the relation with a formula.

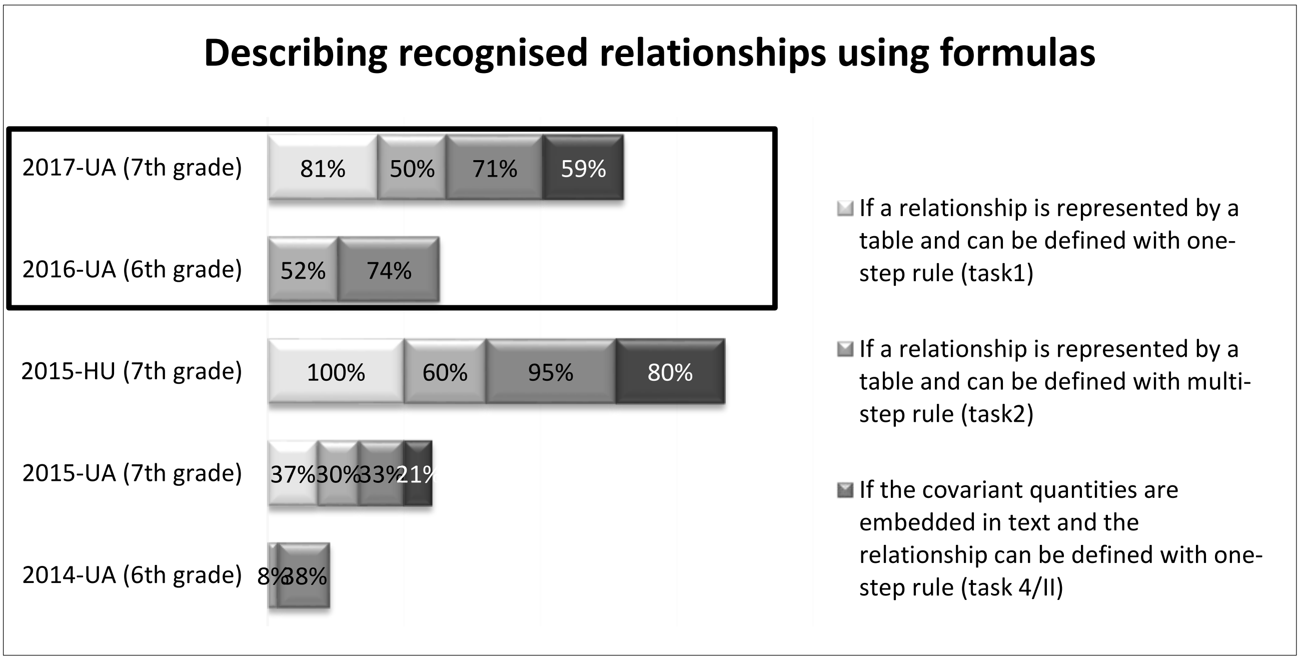


Figure 4

Many of the students used different letters to identify values (e.g.: *s* and *l*, *s* and *x*, *x* and *y*) when describing the relation of covariant quantities with formulas within the context. This suggests that they are „thinking of numbers themselves as variables" (as cited in Rivera, 2013, p. 76.), so the familiarization of the constitutive parts of the variables in the function concept within the program was affected too. Furthermore, teaching different representations of the function concept with 6th grade UA students contributed to their ability to represent one specific problem differently: students produced a formula to express a general solution of a word problem (Task 4) to express a functional relation by recognizing a connection between covariant quantities arranged in a table (Figure 5).

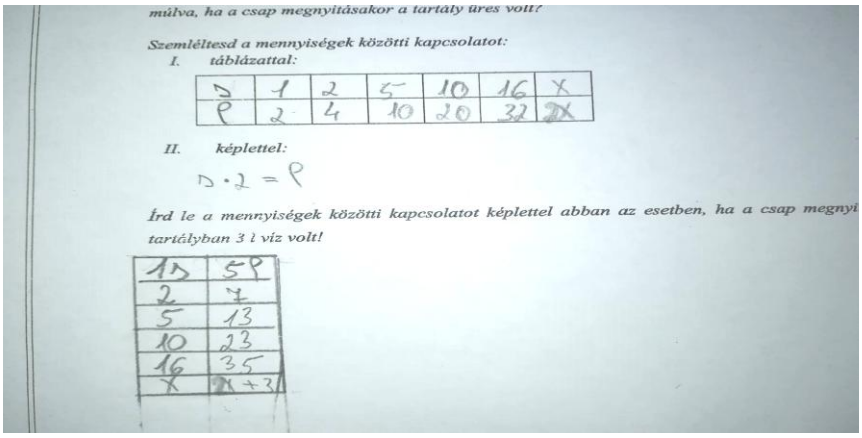


Figure 5

## CONCLUSIONS

This study focused on the effectiveness of the improvement program that aimed the formation of the function concept. We researched whether a 6th grade Ukrainian student group's familiarization with the cognitive roots of the function concept and its different representations affects the formation of the correct function scheme. To answer this question, we used a knowledge-based and skill-based measurement among improvement program participants a year after the realization of the program.

The results show that:

- the results of UA students participated in the program exceeded the results of those 7th grade UA students who participated in the assessment but were not provided with preparatory work;
- the specified preparatory work had a positive effect on the correct formation of function concept;
- the result of UA students participated in the program was close to the result of HU students participated in comparative pedagogical research, for whom the preparatory work started from first grade while the UA students only participated in 11 occasions overall;
- the 11 occasions were not enough for all UA students for them to be able to recognize the relation between differently represented covariant quantities and to support their statement, since we have experienced a downturn in the results of 7th grade participants compared to 6th grade students' results;
- among the measured skills there was no significant change in the ability of formulating the relationship by words.
- However, the effect of the program could be seen in the development of the skills of flexible switching between several representations and of recognizing connections which is necessary for problem solving.

Since the improvement program was completed within the framework of an action research – and its goal is to solve a special problem in a given environment with its result being useful in other practices (Falus, 2004) – our purpose is not to generalize the results. Although they made assumptions which requires further research to be justified: (1) the 6th grade is the age group which needs greater emphasis on skills development regarding the comprehension of function concept; (2) before introducing the definition, the development and extension of the program in 7th grade can be a great help in the formation of the function concept.

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# TEACHING PROBLEM SOLVING IN INCLUSIVE CLASSROOMS

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*In the last few years, the concept of inclusion has become more and more prevalent in mathematics classrooms in Germany. This is not just an opportunity, but also a big challenge for teachers of mathematics. The Friedrich-Schiller-University Jena is a participant in the national programme to improve the quality of teacher training, which is funded by the German Federal Ministry of Education and Research. Within the framework of the subproject „Media in mathematics education", the use of new technologies in the mathematics classroom – especially, but not only digital media – is examined. The main focus in the academic year 2016/17 was to discover how new technologies can help to deal with the heterogeneity of the students, especially with a combination of visually handicapped and sighted students. For some concrete mathematical topics, problem-solving teaching units were worked out according to Pólya. In the paper those are presented and some practical experiences are reported.*

## PROBLEM SOLVING AS A CORE OF (INCLUSIVE) MATHEMATICS EDUCATION

Problem solving is without doubt one of the main activities while doing mathematics, and therefore also in mathematics classrooms. Accordingly, problem solving is one of the six main process-related competences in Germany, which should be developed during school (KMK, 2004: 7 and KMK, 2012: 11). In addition, problem solving is strongly connected to other main activities in mathematics, especially to concept formation (cf. Büchter & Leuders, 2011: 32-35), modelling, reasoning and proving. Heinrich et al. even advance the extreme view that problem solving tasks are not to be distinguished from modelling and reasoning tasks (cf. Heinrich et al., 2015: 282).

Moreover, since the ratification of the UN Convention on the Rights of Persons with Disabilities by Germany in 2009, inclusion is becoming more and more prevalent in German schools, and thus also in mathematics classrooms. The term inclusion means the overcoming of every kind of marginalization, discrimination and social stigma and, particularly in the German discussion about this term, a strong delimitation from the term integration has been enforced (cf. Ziemen, 2015: 101). In school practice, this implied and still implies two consequences. On the one hand, more and more students with different disabilities are going to schools providing general education: Whereas in 2008 17.5% of students with special needs in education were taught in the general school system, which means not in special schools, this ratio was 37.7% in the year 2016 (compare the statistics in: Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2016: 8; Sekretariat der Ständigen Konferenz der Kultusminister der Länder in der Bundesrepublik Deutschland, 2009a: 3 and 2009b: 3). On the other hand, students should not just be taught together in the same classroom (i.e. integrated), but they also should work – as much as possible – on the same tasks and learn in cooperation from each other.

Problem solving can play a key role in inclusive mathematics classrooms, because problem-solving tasks have special properties, which can support the individualization of the learning process. These include multiple possibilities to access a task (e.g. various solving approaches), self-differentiating through free choice of the complexity level of the solution (this aspect is common with the so-called polyvalent tasks, see Sill, 2011) and the possibility of multiple representations. All those properties can make a problem-solving task into an appropriate task in inclusive classrooms: Students can find their own way to visualize, to grade and to solve the problem, but they can also cooperate during problem solving and discuss different ways of solution.

One of the main areas of inclusive mathematics education, in which further research is necessary, is the teaching of sensory disabled and sensory not-disabled students together (cf. „So mangelt es insbesondere an Kenntnissen zu mathematischen Lernprozessen im Zusammenhang mit Sinnesbeeinträchtigungen […]" Korff & Schulz, 2017: 119). For this reason, within the framework of the subproject „Media in mathematics education" at the Friedrich-Schiller-University of Jena a different disability is focussed on in every academic year, starting with visual impairment in autumn 2016. In the subproject, we develop various teaching units for inclusive mathematics classrooms and, if there is a possibility, we also make some practical experiences with those units in school. For inclusive mathematics classrooms with visually impaired and unimpaired students there are already 9 units developed; we will show and discuss two of them in this paper. However, in order to understand why some special changes of tasks, of demonstrations or of students´ tools are necessary, we will give a very short summary of some specialities when partially sighted or blind students do mathematics.

## MATHEMATICS WITH PARTIALLY SIGHTED/BLIND STUDENTS

When doing mathematics with visually handicapped students, it is important to emphasize and to understand, that this is not just about no or limited possibilities to visualize mathematical terms, formulas or relations. Because of the disability, partially sighted and blind students have limited possibilities to make everyday experiences, to explore space (e.g. climbing on a tree, crossing a road) and to make notes. Therefore, they have limited precursor skills in mathematics from primary school, which means in particular that, for them, counting, number and set concept formation, realization of planar or spatial figures is more challenging than for visually not-handicapped children.

Accordingly, instructions in mathematics lessons should be tactile-oriented and acoustic/auditory, students' and teachers' demonstration material should be tactile and well-known from everyday life. Pictures, figures, tables and diagrams can be magnified and simplified for partially sighted students, but they should have a high visual contrast. For blind and strongly partially sighted students, simplified figures etc. with high tactile contrast should be prepared. Digital geometry software is very useful for partially sighted students, especially because of the possibility of magnifying of figures and of changing the settings (thick lines, high visual contrast), but unfortunately it is not applicable if the students are blind. The typesetting software Latex, which requires that formatting information be specified in plain text using special code sequences, can be a bridge language between Braille and common mathematical symbolism because of its linearity and is widely used while teaching mathematics to visually handicapped students (cf. Landesinstitut für Schulentwicklung: 1-8) A detailed argument for the use of Latex can be found in Meyer zu Bexten/ Hiltner, 1999.

The superordinate principle for the selection and modification of students´ material is on the one hand to enable students to operate by themselves on the material and make concrete experiences with it, on the other hand to counterbalance their visual deficits. Furthermore, the used material should be as specific as necessary, but at the same time as normal/usual as possible (cf. Csocsán/Hogefeld/Terbrack, 2001: 299).

## PROBLEM SOLVING TEACHING UNITS FOR INCLUSIVE CLASSROOMS (BLIND/PARTIALLY SIGHTED AND SIGHTED STUDENTS)

Regarding pre-primary and primary school children, it is well known that visually handicapped students are in general cognitively able to solve problems as well as sighted students. However, those competencies can be developed only in connection with content based on concrete perception (Leuders, 2012: 242 – 243). Hence, when teaching mathematics, primarily when solving problems in an inclusive classroom with blind/partially sighted and sighted students, the teacher has to make sure of the following:

- All students have the required knowledge and experiences for solving the problem(s), which means in particular, that also blind or visually handicapped students must have been able to make necessary experiences and to form related concepts.
- The demonstration of the problem(s) is suitable for all students. Particularly, also visually handicapped students get access to the problem(s). This means first of all acoustic and tactile demonstration or visualisation which is adapted (see above).
- All students get suitable tools to work on the problem(s) and these enable them to develop different approaches for solving the problem(s). Similarly, this means acoustic and tactile tools or adapted visual tools.
- There is enough time for all students for understanding the problem(s) and for working on them. Especially visually handicapped students need more time than sighted students for both of those activities.

- All students have the possibility to keep track of the problem-solving approaches and final results. Because of the visually handicapped students, this phase in the lesson should include acoustic, tactile or adapted visual tools, too.

From this it follows that by developing problem-solving teaching units for inclusive classrooms the next three questions can play an important role: (1) Which current problems are already suitable regarding presuppositions and student access possibilities for use in inclusive classrooms? Or rather, how can they be changed into suitable ones? (2) Is the current demonstration/visualisation of a problem suitable for sighted and partially sighted/blind students? Or rather, how can it be made more suitable? (3) Are current students´ tools suitable for both groups (sighted and visually handicapped students), or if not, how can this be rectified considering the above-mentioned principles?

In the next section, we demonstrate two concrete examples of adapting well-known problems to make them suitable for inclusive classrooms, using modifications of the problems themselves, of their demonstration and visualisation, as well as of the related students' tools.

### Teaching unit for solving systems of linear equations

The teaching unit we discuss in this section is based on a problem which was formulated by Pólya (Pólya, 1966: 48):

> A farmer has hens and rabbits. These animals have in total 50 heads and 140 legs. How many hens and rabbits does the farmer have?

Because the problem leads to a system of linear equations, it can be solved by students in the 9$^{th}$ class, so by roughly 15-year-old students. Of course, through considerations with regard to the content, younger students could also find a solution. We assume that students to whom the problem is given already know about proportional mappings, graphic and algebraic solution of linear equations and also the scale model.

The goals in the teaching unit are to develop and to compare problem solving strategies related to systems of linear equations with two equations and two unknowns and to solve systems of linear equations by using at least one of the developed strategies. Additional goals are to describe in the students' own words the connection between the coefficients of the linear equations and the number of solutions, to argue for the correctness of the found connection, to construct examples with no solution through changing the given data, to transfer the developed strategies to systems of three linear equations and three unknowns. We planned a unit of two lessons (i.e. 90 minutes).

Even if the teacher could demonstrate the problem acoustically, meaning that s/he could read it out or tell it to the students, there is no possibility to visualize/demonstrate 50 heads and 140 legs, because the numbers are too big. For a suitable visualization and for activity-orientation in the classroom, which is very useful if also visually handicapped students are present (see above), it is necessary to reduce the complexity of the problem. We suggest a simple reduction by dividing by 10: „A farmer has hens and rabbits. These animals have in total 5 heads and 14 legs. How many hens and rabbits does the farmer have?"

To enable all students to try out ideas through manipulation of concrete materials, it is recommended to use tactile materials for representing the bodies, heads and legs of the animals that have visual as well as tactile contrast. We tried to find tools, which are cheap, easy to obtain in a shop and easy to install. Blank beer mats are very cheap and common in Germany, and are practical to use for representing the bodies of the animals. If we use a black or red felt sheet underneath, a visual contrast is produced (otherwise the beer mats are not really visible on a typical – white or grey – school desk) and also the risk of them slipping away by accident is reduced. To find a useful representation of the legs is trickier: We experienced with matches, which are also very cheap and common, but they do not give enough visual contrast, as Figure 1 shows. Another idea was to use plastic sticks from geometric sets, but they also fail to produce the necessary visual contrast (see Figure 2). The solution we used was to make them ourselves by cutting a foam rubber sheet into narrow longish pieces (Figure 3).

Figure 1: Beer mats on a red felt surface for representing the bodies and matches for the legs – the visual contrast is not high enough

Figure 2: Beer mats on a red felt surface for representing the bodies and plastic sticks for the legs – the visual contrast is also in this case not high enough

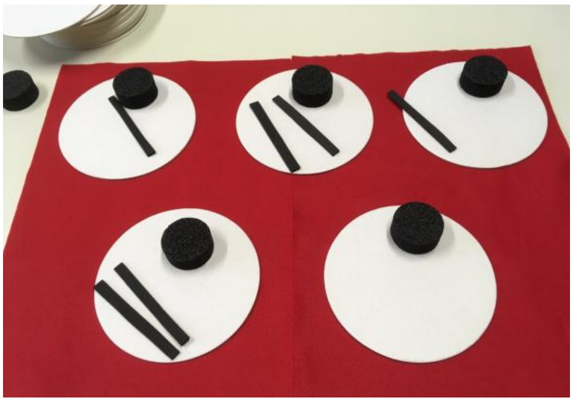


Figure 3: Beer mats as bodies, black round sponge slices as heads and black foam rubber pieces as legs – the visual contrast is high enough

For the teacher´s demonstration we suggest to use a whiteboard or rather an interactive whiteboard. The common blackboards, which are usually green, do not give a high visual contrast. To show a picture on the board that is similar to what the students have in front of themselves, we produced, again from black foam rubber, circles as heads and longish pieces as legs, and glued small magnets on the back side of them (Figure 4).

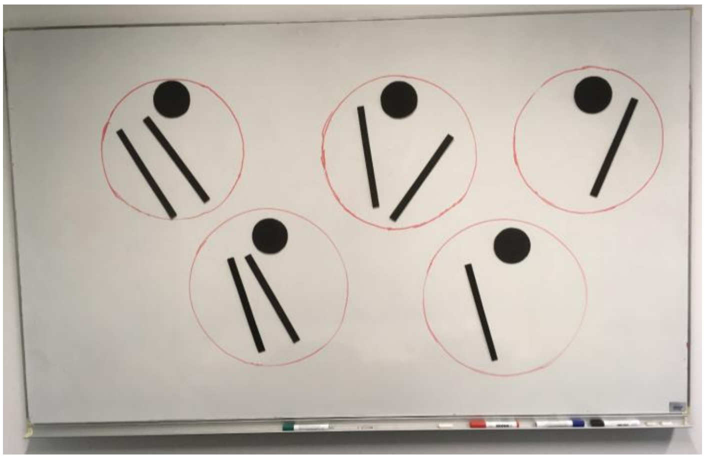


Figure 4: Circles as heads and longish pieces as legs from black foam rubber on the whiteboard for the teacher´s demonstration – the visual contrast is high enough

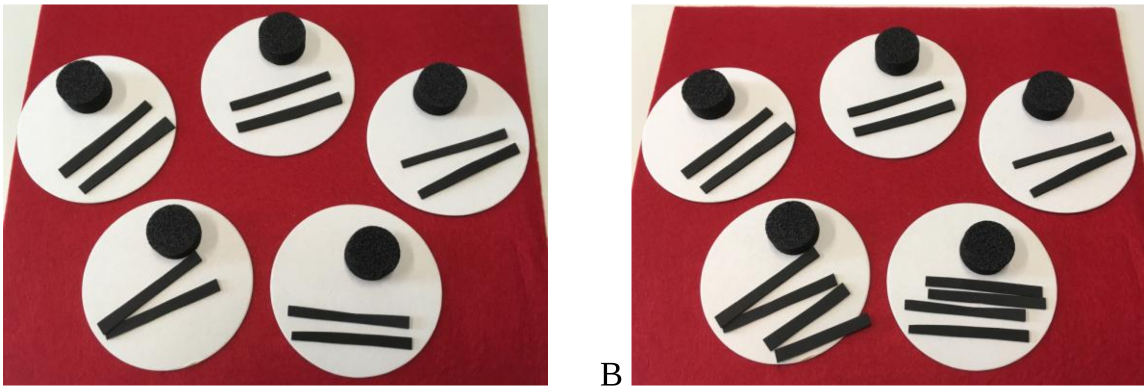
After the students have developed strategies in pairs for solving the problem, a plenary discussion with all students is recommended. We assume the solution does not take too long. The different strategies could be demonstrated and discussed at the whiteboard. We anticipate at least the following strategies:

- Putting two sticks on each beer mat, since every animal must have at least two legs. The remaining 4 legs are mapped in pairs to two beer mats. So, the solution is two rabbits and three hens.
- Putting 4 sticks on each beer mat, meaning in total 20 legs, corresponds to the situation when all of them are rabbits. Afterwards pairs of legs are taken away

until only 14 legs remain. This action is necessary three times to get to 14, so three hens and two rabbits.

- Using equations: h (hens) + r (rabbits) = 5, 2h + 4r = 14. This system of linear equations can be solved through trial and error, through manipulation of the equations themselves, or through considerations with regard to the content.

For example, the first strategy can be carried out by both visually handicapped and sighted students by transforming the starting situation (Figure 5A) into the solved situation (Figure 5B).



A B

Figure 5: Starting situation „each animal needs at least two legs” (A), solved situation: The remaining four legs are mapped in pairs to two heads (B)

Afterwards, solving variations of the problem can give an insight into the deeper structure of systems of linear equations. We suggest to change one given quantity at a time at the beginning, so in a first next step to solve problems like:

- A farmer has hens and rabbits. These animals have in total 6 heads and 14 legs. How many hens and rabbits does the farmer have?
- A farmer has hens and rabbits. These animals have in total 4 heads and 14 legs. How many hens and rabbits does the farmer have?
- A farmer has hens and rabbits. These animals have in total 5 heads and 18 legs. How many hens and rabbits does the farmer have?
- A farmer has hens and rabbits. These animals have in total 5 heads and 12 legs. How many hens and rabbits does the farmer have?

A subsequent plenary discussion of the strategies and solutions is necessary, because the next related problems cannot be demonstrated by tactile materials any more. At this point in the lesson we suggest to give the original problem of Pólya to the students, because they now have enough experience with concrete materials, so they can imagine to make manipulations on heads and legs. For keeping track of the results, tables are useful, as these can be magnified for partially sighted students and tactile ones can be prepared for blind students. We suggest to solve the following problems in the next step:

- A farmer has hens and rabbits. These animals have in total 50 heads and 140 legs. How many hens and rabbits does the farmer have?

- A farmer has hens and rabbits. These animals have in total 500 heads and 1400 legs. How many hens and rabbits does the farmer have?
- A farmer has hens and rabbits. These animals have in total 5000 heads and 10248 legs. How many hens and rabbits does the farmer have?

A discussion and comparison of strategies and results is also useful after this phase in the lesson. Students have to apply their previous strategies for problems on a more complex, but not yet abstract level. In the last phase of the lesson the following problems can be solved:

- A farmer has hens and rabbits. Those animals have in total $k$ heads and $l$ legs. How many hens and rabbits does the farmer have?
- Under what conditions does the problem have no solution?

By facing these problems, the students eventually realise that the strategies involving calculating up or down are not efficient. On the other hand, using linear equations to describe the connections between the data is fast and accurate. The solution of the first question in this last phase is also relevant for the solution of the second question: If a solution exists, $2k \leq l \leq 4k$ should be satisfied. Besides, the parity of the unknowns should be respected.

We also formulated an analogous problem with three equations and three unknowns for the next sessions in school: A farmer has hens, horses and rabbits. Those animals have in total 5 heads, 16 legs and 60 toes. How many hens, horses and rabbits does the farmer have? A possible representation of the toes can be carried out by using glue dots with a high visual contrast, so for example red dots on black foam rubber pieces as legs (Figure 6). This kind of representation can enable not just partially sighted students through the high visual contrast to participate in the problem-solving process, but also blind students can recognize the tactile difference between the beer mats, the foam rubber pieces and the glue dots. Therefore, they can participate in the problem-solving process, too.

Figure 6: representation of the solution of the generalized problem – glue dots demonstrate the toes of the animals

There are no practical experiences related to this teaching unit yet.

**Teaching unit for geometrical probability**

The second teaching unit we present in this paper is based on a classical stochastic problem, well-known since the 18th century as the game ´franc-carreu´ (Gnedenko, 1997: 410ff) and found in almost every textbook related to stochastics. We formulated the following variation: „Anne and Bob are flipping a 1-cent coin onto a grid. Anne wins the cent if the coin does not touch any line, otherwise Bob wins it. Is the game fair?"

The students facing the problem do not need to have too much previous knowledge. We assume, they know already about frequentist and classical (i.e. Laplace-) probability, probability of opposites and besides about some basics in geometry (properties of squares, circles, parallel lines). If they have already learnt about expected value, this can be helpful, but it is not absolutely necessary. We planned this unit for students in the 11th class, so for roughly 17-year-old students, but the problem is of course discussable much earlier, maybe from the 7th class (12-13-year-old students) up. The goals of the teaching unit are to develop and compare problem solving strategies related to the coin-flipping problem and to calculate probabilities by using diverse problem-solving strategies and properties of geometrical objects. Besides that, to define the mathematical term „fair game" by using everyday language, and to develop and justify different strategies to make the coin-flipping problem into a fair game. This unit also comprises two lessons (i.e. 90 minutes in total).

It is clear, that by this unit even if there are no visually handicapped students in the classroom, the students need to try out the game, which means, they need a grid and a coin to flip in pairs. It is also obvious, that the size of the grid and of the coin play an important role: If the coin is smaller, Bob has less chance to win etc. At some point, the students have to measure the distance between the lines of the grid and the diameter/radius of the coin. Because for visually handicapped students measurements in the millimetre range are not really possible (the smallest possible distance they can recognise using touch is 0.5 cm), we magnified all given data. In place of the 1-cent coin (diameter ≈ 1.1 cm) we suggest using a 2-euro coin (diameter ≈ 2.5 cm). The original grid had a line distance of 2 cm; we changed this proportionally into 5 cm (figure 7).

For visually strongly handicapped and even blind students we prepared a tactile grid on a special drawing board. Based on practical experience, we also figured out that flipping the 2-euro coin on this tactile grid is not good enough yet: While trying to find the coin´s place on the grid by feeling around, we usually moved the coin away, which makes the stochastic problem pointless. So, because the board has a metal under surface, we tried out flipping magnets, just very strong ones (available with the diameter of 2.5 cm) were strong enough not to be pushed away. In Figure 8 below, the prepared tactile grid is shown.

Figure 7: Drawn grid with 2-euro coins for the coin-flipping problem

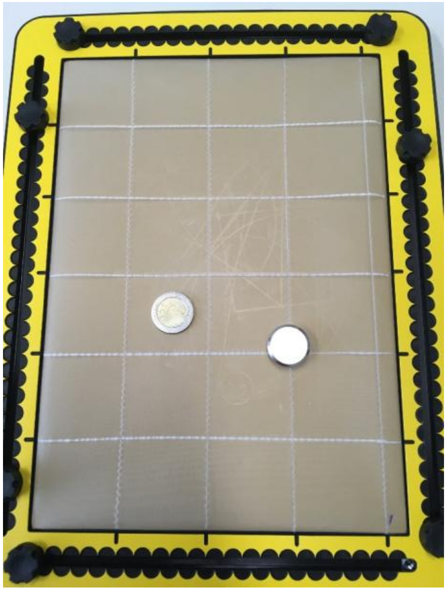


Figure 8: Tactile grid with strong magnets on a special drawing board for strongly visually handicapped students. Students´ tools for the coin-flipping problem

In the developed teaching unit, we suggest to start with a plenary discussion about fairness in everyday life, in sport etc. Students can freely talk about their ideas and beliefs about fairness in different areas of life. The next question the teacher should ask is: How should fairness, and a fair game, be defined for mathematics? For example, if two people play a certain game, what criteria should be satisfied, if those people would like to play a fair game? The list of criteria doesn´t have to be completed and can be discussed at the end of the unit.

After this introduction, the teacher explains the coin-flipping problem, which can be done just acoustically, and suggests to try the game out. The prepared students´ tools can be set up. The students may decide themselves, how to approach the problem. Because they already have experiences with classical and frequentist probability, they can use both of those concepts independently. (For example, flipping the coin 100 times and seeing if one of the outcomes happens more frequently, or measuring and comparing possible winning areas on the grid). Working in pairs is recommended.

After roughly 15 minutes in a plenary discussion the ideas and problem-solving strategies should be demonstrated and discussed. For this phase of the unit, a teacher´s demonstration of the problem on the board should be prepared. We suggest using two flipchart sheets and a cardboard circle (Figure 9) attached by magnets onto a blackboard. Black lines on a white sheet have a stronger visual contrast than white lines on a common blackboard.

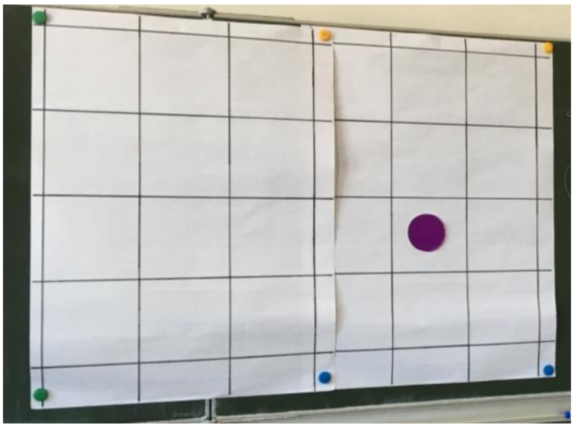


Figure 9: Flipchart sheets and cardboard circle for teacher´s demonstration of the coin-flipping problem

Depending on the quality of the ideas in the discussion, or also in the working phase, if students cannot approach the problem at all, an analogous problem with less complexity can be offered: „Anne and Bob are flipping a 2-euro-coin onto a sheet with parallel lines at a distance of 5 cm. Anne wins the coin if it does not touch any parallel line, otherwise Bob wins it. Is the game fair?" Drawn and tactile sheets should also be prepared for this problem (Figures 10 and 11).

Figure 10: Sheet with parallel lines and 2-euro-coins for the analogous problem

Figure 11: Tactile sheet with parallel lines and magnets for the analogous problem

The main point of the problem-solving process is to recognise that whether or not the lines are touched depends on the position of the coin, which can be described through the position of the centre. To direct students´ thoughts in this direction, the analogous problem with the parallel lines can help. In this case it is very simple to find the areas in which the coin can land advantageously for Anne/Bob, because those areas are bounded by parallel lines to the originally parallel lines. Figure 12 shows the areas advantageous for Anne (left) and for Bob (right) in the original problem (advantageous, if the midpoint of the coin is in that area).

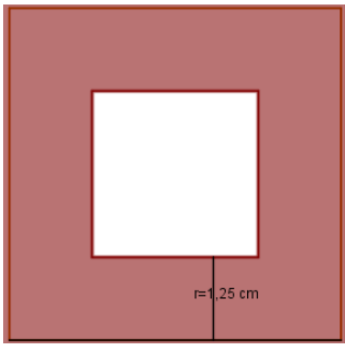



Figure 12: Areas advantageous for Anne (shaded) and Bob (unshaded) (figures not true to scale!)

This should be worked out in an additional plenary discussion, which also leads to the insight that Bob has less chance (25%) to win than Anne (75%).

In the next step of the unit, the teacher offers a similar, but slightly different problem to solve: What if Anne only wins if the coin touches both lines? After the approach previously discussed, it should not take too long for the students to solve this problem. Figure 13 shows the area advantageous for Anne in this case.

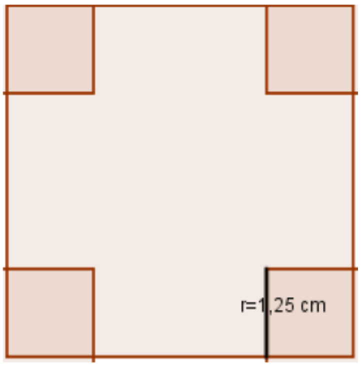



Figure 13: The darker shading shows the area advantageous for Anne in the modified problem (figure not true to scale!)

We assume, students will figure out quickly, that in this case the game is fair (both have the same chance to win the coin). The teacher can ask them what other possibilities there are to make the original problem into a fair game. Students can develop ideas, at first in pairs for a few minutes, which should be discussed additionally in a plenary session. For example: To magnify the grid, to reduce the size of the coin, to change the amounts won by each player, to change the advantageous areas (the last problem is already a possible realization of this idea). In closing the lesson/unit, the teacher can suggest to go back to the list of criteria of a mathematical fair game and complete/modify it.

This teaching unit was already tried out one time in school practice. In the next section we report our experiences. These can be especially interesting and useful for didactical research, because other experiences with visually handicapped students when solving problems are not known yet, at least to the author of this paper.

**PRACTICAL EXPERIENCES**

The teaching unit for geometrical probability discussed above was applied in Jena, Germany in a public comprehensive school in June 2017, in the last week of the school year. The class consisted of 12 students at the age of 17-18 years, including one visually handicapped female student. She experiences occasional double vision, and her field of vision is limited. She does not receive any additional help from the current teacher besides magnifying some figures and tables. If she cannot see something on the blackboard, she asks her neighbour. The class already had experience with frequentist and classical probability, with probability of opposites, with contingency tables and stochastic independence as well as with expected values and standard deviations, and also of course with basics in geometry. So, all the necessary preconditions were satisfied. For the teaching unit, we used a double lesson (90 minutes). Three university students who participated in the project, the current mathematics teacher of the class and myself joined the class as advisors, but just one of the university students acted as teacher. All of the others joined small groups during teamwork and gave – if they were asked – advice to a group of 2-3 students at the same time.

The teaching unit was carried out as planned: After a short discussion about fairness in everyday life, students tried to give a definition for a fair game in mathematics. After the teacher (one of the university students) introduced the flipping-coin problem and gave the prepared gridded sheets to the students, they started to work on the problem. Because the students could not find an approach for a while, the teacher switched to the less complex problem with the parallel lines and spread the new sheets. After that, the students started to work on the problem in groups of 2-3 people. During this phase of teamwork, the special drawing board and the tactile sheets were offered to the visually handicapped student. She was obviously uncomfortable with this and refused indirectly to work with the drawing board. She also reported being unable to keep the problem in mind. So, she went back to her group without the special board and tried to listen to other students´ conversations about solving the problem. She did not draw anything during the whole lesson either with a pencil or tactilely, and did not really participate in the discussions, just listened. It was clear that she felt more comfortable when not receiving special attention from the teacher.

In summary, even if the teaching unit was successful, students developed different approaches to solve the problems (the original and the less complex one), eventually found the right solution and also developed different ideas about how to make the game fair, we have to admit that the developed inclusive aspect did not work. We cannot yet conclude with certainty that the preparation was insufficient or inappropriate, because the visually handicapped student did not even try to work with the special device. In order to decide this, further practical experience is necessary. What we have found instead is something else.

## CONCLUSIONS

Based on the fact that the visually handicapped student refused to work with special tools in the mathematics classroom, which was a very interesting and surprising experience, we can step back and ask, why this happened. Of course, there were four new advisors in the classroom, but those were new for all of the students and the sighted students worked on the sheets they got without hesitating. Also, just one of the new people in the room acted as a teacher, the other advisors were there only to provide help if help was asked for, and did not direct conversations or give suggestions for solutions. The main difference was that the visually handicapped student briefly received special attention and a special device to work with.

Our hypothesis is that this student found the extra attention to be such a negative experience that it overwhelmed any improvements in mathematical understanding or problem-solving ability the special device may have offered her. She tried to behave like students without a handicap, listened to conversations for catching up on some ideas, perhaps feeling that this would be sufficient. So, on a more general level, there is a contradiction between the special needs (and skills) of handicapped students on the one hand and the wish to avoid getting special treatment or attention on the other hand. Therefore, the main conclusion from this experience is that successful inclusion requires more than „just” teaching in a way that accommodates students’ varying levels of physical and mental ability: it must also overcome the social and psychological barriers to being seen as „different”. Particularly during the teenage years, when many students feel pressure to fit in, this is in some ways a more difficult and fundamental problem to solve.

My special thanks go to the students participating in the project in the summer term 2017: Tobias Allerdt, Lukas Leistner and Daniel Gaschler.

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# IMAGINARY INTERVIEW WITH ANDRÁS AMBRUS

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**Abstract.** *„The Life's Time for Mathematics Education and Problem Solving" is the title of the Festschrift edited by Martin Stein on the 75th birthday of András Ambrus ([30]). This activity should be also mentioned in the row of articles of ProMath 2017. This imaginary interview is my personal greeting, based on 40 years common work with András Ambrus. The goal is to introduce some snapshots of András Ambrus' long and successful work.*

*We focus on those tools, methods and content of mathematics teaching and learning which can make the experience of successful problem solving available to pupils with average interest and ability, while also being useful in developing problem solving skills of talented pupils.*

## INTRODUCTION

András Ambrus as a teacher, an author, a supervisor has a significant impact on mathematics education, mathematics teacher training, research on didactics of mathematics and researcher education in Hungary. He is the author of the first textbook in Didactics of Mathematics. Through his personal international contacts, many Hungarian researcher of mathematics didactics didactics have been involved in international scientific life. As a founding member, he promoted the introduction of Ph.D. program on didactics of mathematics in Hungary. On his 75th birthday he was greeted by disciples and honoraries with an English and a Hungarian festive edition ([27], [30]).

We have been working – sometimes fighting – for the same goals since 1977. Although he was not my teacher and my boss was for a short time, I learned a lot from him. Over the years we have travelled together thousands of kilometers to international conferences by car, so we had a great opportunity to discuss the things of the world. I base my next imaginary interview on these conversations and try to introduce some snapshots of András Ambrus' long and successful work ([7], [8], [10], [11]).

## QUESTION 1

*You have been a successful and respected mathematics teacher in Kiskőrös. Why did you leave your secure position for studying at ELTE University in Budapest?*

AA: I loved my students and mathematics as well, but I wanted to understand them better and to know more mathematics. In Budapest I learned pedagogy and psychology, didactics of mathematics. I found preparing the future teachers interesting, however I did not stop teaching at school. I completed my studies, made my educational and research work, took part in many conferences, I met and kept personal contact with leading researchers. That inspired me to start an intensive research period in Halle with my supervisor Professor Werner Walsch ([4]).

Figure 1. Visiting Professor Werner Walsch.

**QUESTION 2**

*You are involved in care of gifted students. Do you think, that there is too much attention on the gifted students in Hungary?*

AA: I would like to emphasize that developing the talented is very important in the mathematics education. We can most certainly assert that the education of the gifted in Hungary in mathematics is world-famous. There is a great chance that we will not lose any talents in mathematics. Since 1894 we have the Mathematical Journal for Secondary Schools (the second in Europe!). This journal organizes a continuous competition with very demanding tasks. There are also many (in my opinion too many) mathematical competitions (even in grade 3). I think that even today there are enough wise people in Hungary to deal with the promotion of gifted students in the mathematics education.

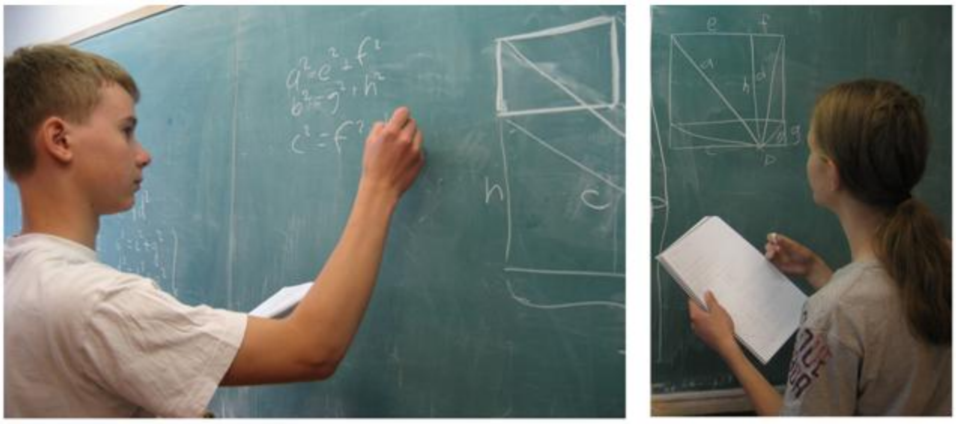


Figure 2. Working with gifted students.

I'm more interested in the remaining 90% of the students. ([5], [6], [9], [15], [16], [17], [18], [23], [24], [25]).

## QUESTION 3

*As we know your main research topic is the development of the problem solving ability. Do you mean to teach problem solving to the 90%? What is the focus of this kind of teaching?*

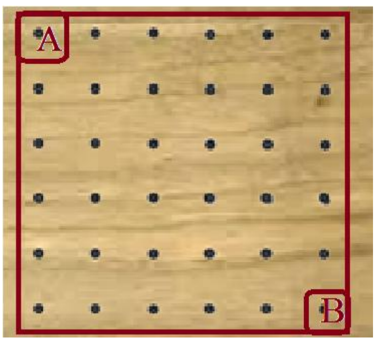
AA: There are some needs and possibilities to change the style of mathematics teaching. The mathematics textbooks and problem book contain only closed problems. Most of the mathematics lessons are strongly teacher centred, the use of pair or groupwork is not characteristic. The use of open problems is very uncommon in Hungarian mathematics teaching. We carried on several experiments with my doctoral students in this direction. We successfully used visual and concrete representations, open problems and problem variations, methods of creating new mathematical problems by association, analogy, generalization, cooperative methods and worked-out examples ([1], [2],[3], [12], [13], [14], [19], [20], [21], [22]).

We prefer problems with multiple possible solutions for which different mathematical knowledge on different levels of abstraction is necessary. For example we consider the following versions of the same problem.

a) Mark a 6 by 6 rectangle on the peg board. How many possibilities are there to go from the upper left corner (*A*) to the lower right corner (*B*) if you can make one step at the time, and that either to the right or down.
b) How many different ways are there to read the word „MATHEMATICS" in the table if you can read adjacent letters from left to right and top to bottom.

We can count the number of moves on the peg board or on the table directly or we can connect the problems and use the Pascal triangle for all problems of similar type.

a)




b)

| | | | | | |
|---|---|---|---|---|---|
| M | A | T | H | E | M |
| A | T | H | E | M | A |
| T | H | E | M | A | T |
| H | E | M | A | T | I |
| E | M | A | T | I | C |
| M | A | T | I | C | S |

c)

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 3 | 6 | 10 | 15 | 21 |
| 1 | 4 | 10 | 20 | 35 | 56 |
| 1 | 5 | 15 | 35 | 70 | 126 |
| 1 | 6 | 21 | 56 | 126 | 252 |

Figure 3. The peg board, the table and the 6 by 6 rectangle on the Pascal triangle ([28]).

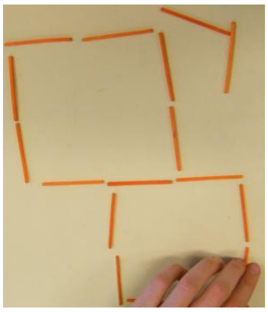
## QUESTION 4

*What factors influence the way of posing a problem?*

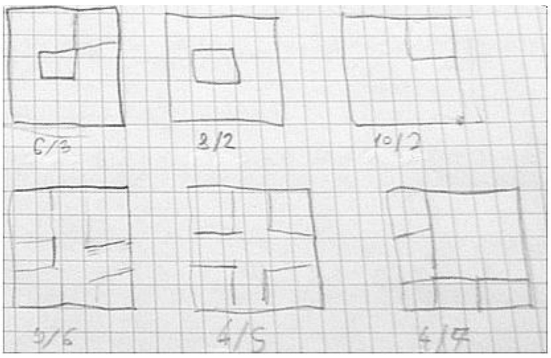
AA: There are *subjective factors* like age, previous knowledge, experiences with the content of the task and *structural factors* like introductory, practicing, transferring to the same topic, transferring from one topic into another topic. By transferring we need to take into consideration that some knowledge might have been learnt some weeks (months, years) before. One of the problem posing factors is the type of the problem (closed, open).

A good *introductory problem* is, for example, the matchstick game: we build a 3 by 3 grid of 24 matchsticks. A certain number of matchsticks should be taken away such that the remaining matchsticks form a given number of squares. (All the squares are counted and each matchstick must be part of a square.) The game can be played manually, with pencil on the square grid paper or on a computer ([26], [29]).

a)



b)



c)

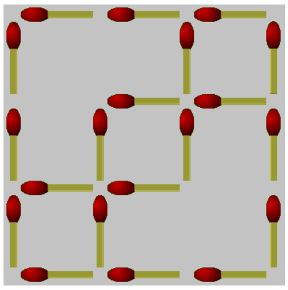


Figure 4. The matchstick game played manually, with pencil and on computer.

Another good example is the problem of summing the first $n$ odd positive integers, which can be proposed by *different formulations*.

a) What is the area of the big triangle if the small triangles have unit area?

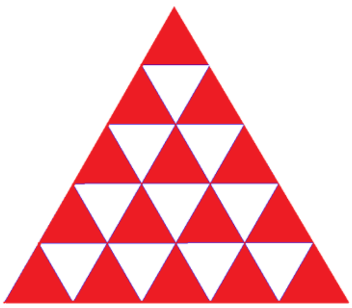


Figure 5. Determine the area of the big triangle

b) Show that the sum of the first $n$ odd numbers is a square.

c) Determine the sum of the first $n$ members of the arithmetic sequence, with the difference 2 and starting with 1.

d) Show that the difference of two consecutive square numbers is always an odd number.

e) There are 7 chairs in the first row of a trapezoidal auditorium, and there are two chairs more in each row than in the one in front of it. How many seats are in the auditorium if the last row has 21 chairs?

*There are different ways to solve these problems* (arithmetic-algebraic, visual solutions). There are several possible *strategies* even in geometric representations: looking for the sum directly (Figure 6), for the fourfold of the sum (Figure 7), or for the twofold of the sum (Figure 8 and the figure on the cover of this book).

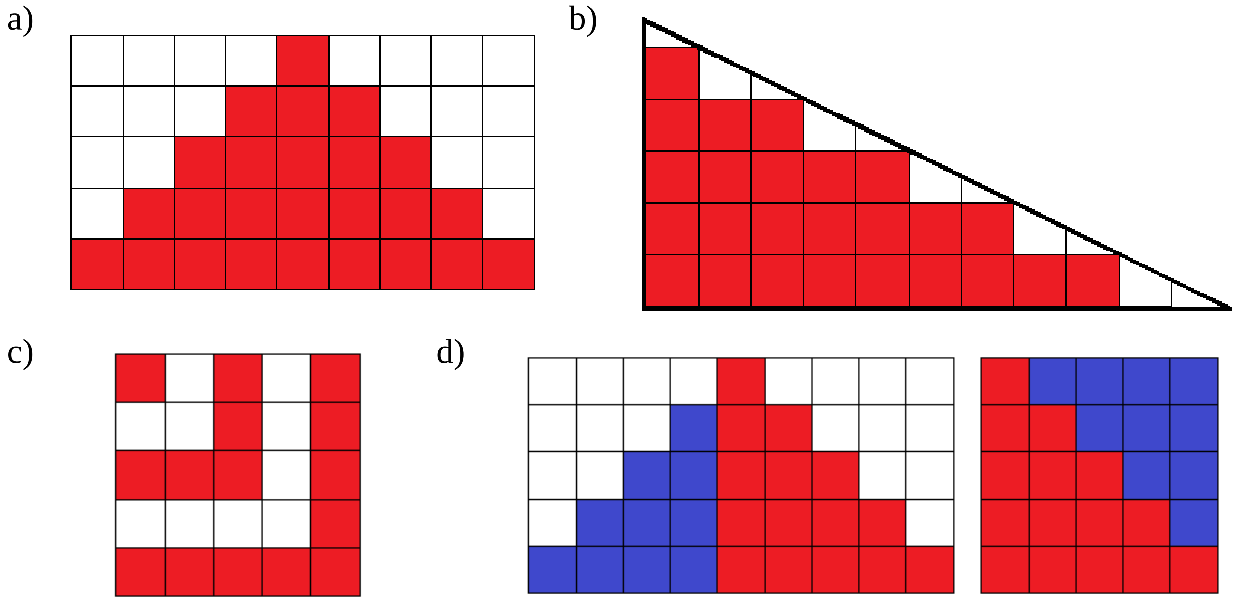



Figure 6. Looking for the sum directly.

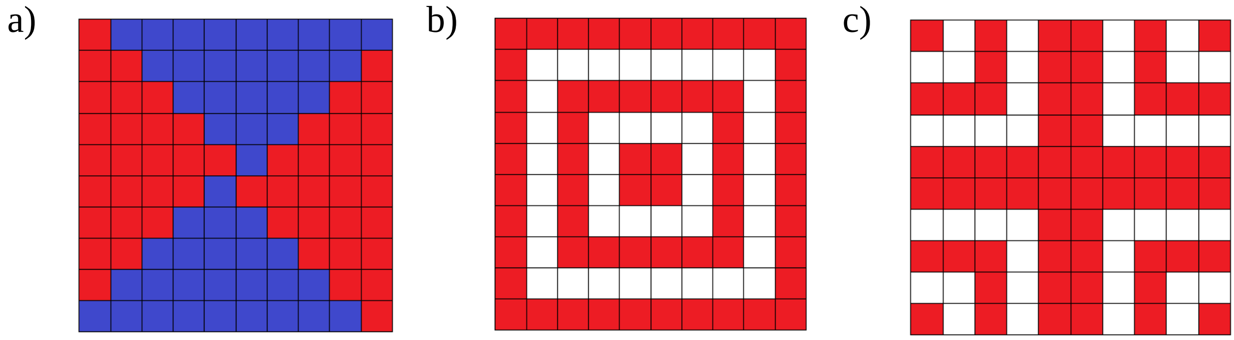



Figure 7. Looking for the fourfold of the sum.

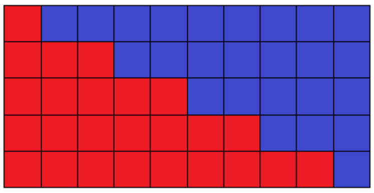


Figure 8. Looking for the twofold of the sum.

**QUESTION 5**

*What do you think about the role of the teacher during the problem solving process?*

AA: Based on my half century mathematics teaching experiences I'm convinced that the guidance is very important for most of the pupils. The level of guidance strongly influences the achievement of the children. We may pose the problem and let them solve it individually or in groups *without any help* (Figure 9). We can guide the solution with the help of *directing questions, prompts* (Figure 10). At quite new and complex problems the teacher might want to *explain the whole solution* step by step with the mathematical argumentation behind it (Figure 11).

Figure 9. Problem solving without any help.

Figure 10. Problem solving with the help of directing questions.

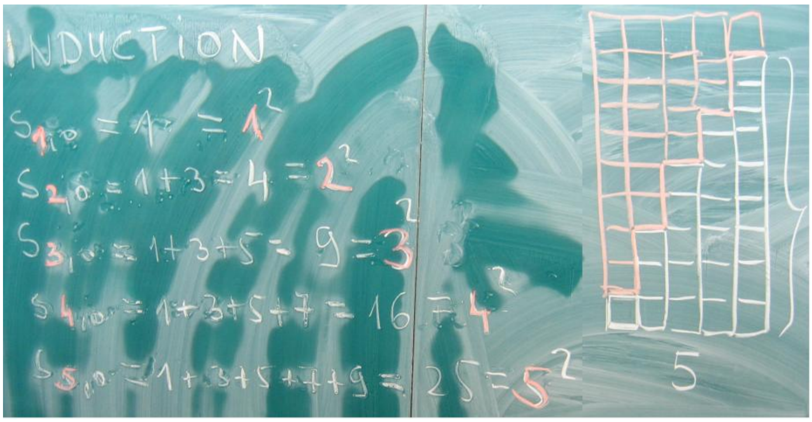



Figure 11. Explaining the whole solution.

**QUESTION 6**

*In textbooks and problem books there are many tasks similar to the following: Factorize the polynomial expression* $a^4 + 4$. *What can the 90% do with this problem? Do you think that this kind of problems are useful for them?*

AA: Of course, such tasks can be dealt with in the direct treatment of the topic „algebraic transformations, identities”. Then the students think quicker of the familiar formulas, rules. We can use it as a guided problem. The method of solution and the level of guidance depends on the situation and the actual topic.

***Situation 1: The students are not familiar with complex numbers, but they know some algebraic identities.***

We add $4a^2$ to the expression and subtract the same:

$$a^4 + 4 = a^4 + 4 + 4a^2 - 4a^2 = a^4 + 4a^2 + 4 - 4a^2.$$

We change the point of view and recall the formula for squaring a sum of two terms:

$$(a^4 + 4a^2 + 4) - 4a^2 = (a^2 + 2)^2 - 4a^2.$$

We change the point of view again, we consider it as the difference of quadratic expressions and factorize: $(a^2 + 2)^2 - 4a^2 = (a^2 + 2 + 2a)(a^2 + 2 - 2a)$.

***Situation 2: The students are familiar with complex numbers and algebraic identities.***

We consider the fourth roots of –4: $1 + i$;  $1 - i$;  $-1 + i$;  $-1 - i$.

Using the polynomial factors of degree one:

$$x^4 + 4 = (x - (1 + i)) \cdot (x - (1 - i)) \cdot (x - (-1 + i)) \cdot (x - (-1 - i)).$$

Using two conjugated pairs of them we get two polynomials of second degree:

$$\big(x - (1 + i)\big)\big(x - (1 - i)\big) = \big((x - 1) - i\big)\big((x - 1) + i\big) = (x - 1)^2 + 1 =$$
$$= x^2 - 2x + 2 \text{ and}$$
$$\big(x - (-1 + i)\big)\big(x - (-1 - i)\big) = \big((x + 1) - i\big)\big((x + 1) + i\big) = (x + 1)^2 + 1 =$$
$$= x^2 + 2x + 2.$$

The solution is: $(x^2 - 2x + 2)(x^2 + 2x + 2) = x^4 + 4$.

***Situation 3: Using Computer Algebra System.***

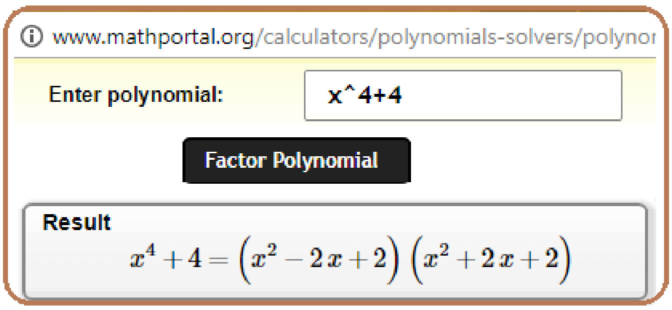



Figure 12. Solution of the problem in Question 6 with the help of CAS

The necessary competencies are different from the classical ones:

• we have to find a program www.mathportal.org,
• we type in the term (with the right syntax),
• we confirm the command,
• we read and interpret the result.

It is worth showing – even with using CAS – an example when there is no real roots, but it is possible to factorize the polynomial over real numbers.

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