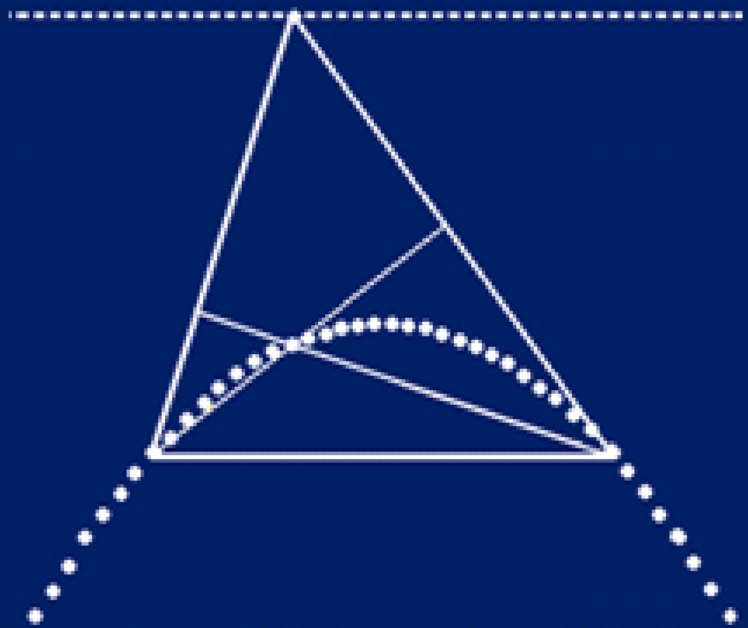
András Ambrus
Éva Vásárhelyi (Eds.)

Problem Solving in **Math**ematics Education



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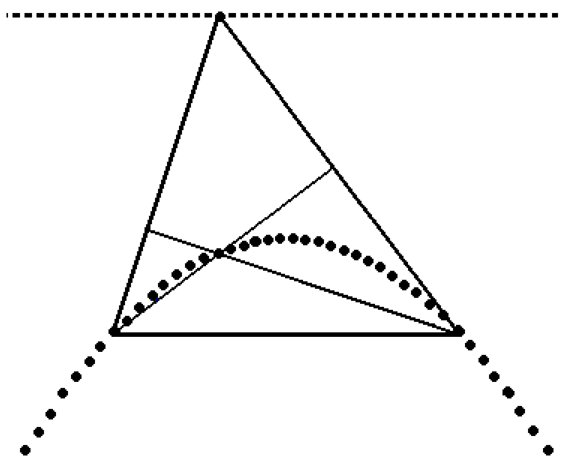

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EÖTVÖS LORÁND UNIVERSITY
FACULTY OF SCIENCE
INSTITUTE OF MATHEMATICS
MATHEMATICS TEACHING AND EDUCATION CENTER

Problem Solving in Mathematics Education

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Content

Preface

ProMath (**Pro**blem Solving in **Math**ematics Education) is a group of didacts of mathematics from all over Europe, who have the common aim of furthering and scientifically exploring problem-solving activity in mathematics among students, exploring the possibilities and preconditions of problem-solving orientation in mathematics teaching, and promoting it.

ProMath was founded by Günter Graumann (University of Bielefeld, Germany), Erkki Pehkonen (University of Helsinki, Finland) and Bernd Zimmermann (University of Jena, Germany). One of the activities of this group is to organize the annual conferences since 1999.

It is a great pleasure and honour for us to organize and support the 11th ProMath Meeting in Pólya's home country.

Since Pólya wrote his famous books, the ideas expressed in them unfolded an impressive effect at home and abroad as to following topics:

- dealing with gifted children,
- problem oriented teaching of mathematics,
- mathematical problem solving for all,
- connecting general problem solving competencies with mathematical problem solving,
- incorporating teaching of problem solving into teacher training,
- etc.

You can find this various aspects in school-reality, in didactical researches, and within the thematic of the conference and of this book.

One of the pedagogical talents of Pólya can be seen by his art posing a problem. He addresses different levels of comprehensive capacities and experiences.

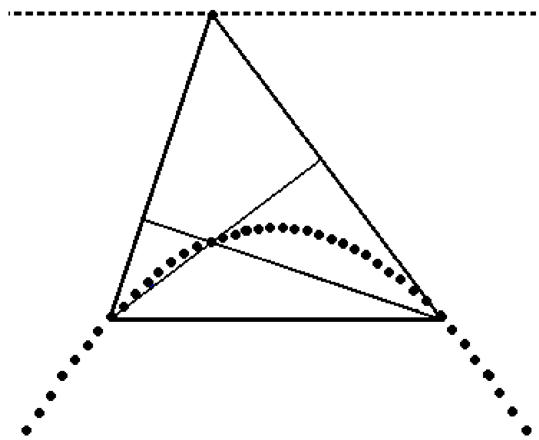
A typical question in the sense of Pólya:

> *A vertex of a triangle moves along a straight line/a plain, the two other vertices of the triangle are fixed.*
>
> *We are looking for the trace of the orthocenter of the triangle.*

It depends on the experiences, knowledge, and other individual aspects of the person which problem – if at all – is derived from this question.

This problem seems to be an inner mathematical one, but it can also be seen as an intermediate step of a real modeling or an analogous procedure:

A central question of mechanical engineering is to describe the useful or disturbing movements of a construction. In order to be able to find a forecast with help of mathematics, we turn to the mathematical model of the construction and try to determine the potential positions of the proper elements (with suitable methods). Very often the still more difficult part of the task is to apply the model-referred results to reality and to describe the range of validity. In this sense we can say, that the „search for the locus" problems are tasks of a modeling process.



The figure on the front page is an illustration of the solution (a parabola) for the case, if the movement area of the vertex is a straight line lying parallel to the fixed side of the triangle.

This volume contains the papers regarding to the talks wich was given during the meeting. The papers of this volume are peer-reviewed according to the contents, organized by András Ambrus. For the quality of the usage of English language each author is responsible for himself/herself.

I want to thank all participants for their valuable contributions as well Mathematics Teaching and Education Center of the Eötvös Loránd University for supporting this volume.

Budapest, 2010.

Éva Vásárhelyi

Hungarian teachers' opinion about problem solving teaching in mathematics

András Ambrus

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Abstract

The Hungarian mathematics education is famous concerning its problem orientation. Based on 12 teacher responses we investigated four questions: influence of Pólya, teaching to think, about the methods of problem solving and developing clever, diligent but not high ability students.

Keywords

Pólya problem solving phases, fostering of talented students, teaching methods for problem solving, fostering the least able students

Introduction

Solving complex, challenging tasks (problems) is in the centre in Hungarian mathematics teaching. Interesting is to mention that we in Hungary rarely use the word "problem", instead we are speaking about solving complex, demanding tasks. Looking for the international situation, our complex, demanding tasks fulfil the conditions of the criteria of problems, so we may use the problem solving terminology to characterize our mathematics teaching. In the following study I will analyse some mathematics teachers' responses relating to problem solving teaching. I use an interview book in which 10 excellent mathematics teachers explain their views about mathematics teaching (GORDON, 2007). A real problem is that from these 10 teachers 8 are coming from Budapest and only 2 from country side, although 80% of all Hungarian inhabitants are living in country side. I asked 13 experienced, good mathematics teachers – all are working not in Budapest - to write down their experiences in the problem solving teaching. Until now I received 6 responses only. It seems that the oral interview is more effective then the written one.

Additionally I can cite G. Pólya, he visited Hungary in year 1971 and gave a lecture for teachers additionally he hold a mathematics lesson in a secondary school too. (PÓLYA, 1971) Most of the interviewed teachers emphasized the importance of the mathematics competitions. They have an old tradition in Hungarian mathematics teaching. The success on the International Mathematics Olympiads is very important, but it concerns only approximately 0,1 percentage of a year generation. Numerous regional and countrywide mathematics competitions help to foster the talented pupils. The fostering of talented students influences in a great manner the whole Hungarian mathematics education. The teachers tend to put the competitions problems into their mathematics teaching, because their works are evaluated after their success on the competitions. These new style problems are embedded in the new mathematics textbooks too. We are much more interested for the remaining part, but for this part of students the competitions problems usually are very hard.

A brief summary of the teacher interviews: The most important part of Hungarian mathematics education since 100 years is the task solution. The tasks are mostly pure mathematical types. By the opinion of the mathematics teacher community those tasks (problems) are valuable which solutions need the application of some special thinking operations or principles taken from different topics. The teacher chooses such problems with help of their solutions the students can acquire the teaching material. (Problem centred mathematics teaching) The good mathematics textbooks and teacher books are written in this sense. The difference between the styles of the mathematics lessons depends from the teachers, in what manner are they able to activate their students with help of tasks, problems, how they empower their discoveries.

In our article we analyse 4 characteristics of Hungarian problem solving teaching: *Using the Pólya phases, teaching to think, about the teaching methods, fostering of the not highly talented but clever, diligent students.*

Using Pólya's problem solving phases and questions

In the interview book (see above) only 2 teachers mentioned directly Pólya, from other six teachers, only four.

1. I think it is indispensable for the future and practicing teachers to know the following books from Pólya: *"How to solve it. A new aspect of mathematical method."* and "*Mathematical Discovery. On understanding Learning and Teaching Problem Solving."* From these books they can learn how it is possible to develop the thinking abilities of their students. On my mathematics lessons in the secondary school the four problem solving phases: *understanding the problem – devising a plan – carrying out the plan – looking back* with all of Pólya's questions were put up on the blackboard. While the student solved the problem at the blackboard (typical in Hungary), he (she) posed questions for himself (herself) based on Pólya's questions in the different problem solving phases. This event helped the students to develop their thinking ability. One student of mine has written to

me: “I – as a medical doctor researcher –am thinking always in my work based on Pólya’s questions” (RÁBAI, 2007)

2. Pólya visited Budapest in 1967 and gave a mathematics lesson in a secondary school and a talk for mathematics teachers. It was a great event for us, for mathematics teachers to see how the world famous scientist activated the students with interesting problems. After the lesson he gave a fascinating talk about the problem solving teaching. Pólyas two books – How to solve and Mathematical Discovery are well known in Hungary between mathematics teachers.

   Pálmay cites Pólya’s ideas about the strength in mathematics education what he told on his Budapest talk in 1971: “ It is awful that in mathematics teaching in USA for 12 year old students was introduced the axiomatic(formal) method. For example it was compulsory to prove: *If three points are lying on a straight line, from these three points there is exactly one between the other two points.* Tough Euclid has proven the triangle inequality (the sum of the length of two sides in a triangle is greater then the length of the third side), it is superfluous to prove it in the secondary school. The students do not understand it, because they think it is obvious, the dogs know too, that if there is a piece of bacon some meters far from him, he goes straight away to it, not on a detour. Of course we need to prove for example the cosine theorem.”

   Interesting to mention that in Hungarian mathematics teaching there is a tendency to prefer the informal, flexible, creative solutions instead of the formal ones in the sense of Pólya. (PÁLMAY, 2007)

3. During the problem solving process it is possible to acquire Polya’s phases. In reality we apply at solving one problem usually only 4 – 5 Pólya questions. Until now I did not think consciously for Pólya’s questions, but I plan to give the whole list to different groups trying to solve mathematical problems. I think it will help them in their work. (KELEMEN, 2008)

4. The most important book for mathematics teaching is in my opinion Pólya’s “How to solve it”, first of all its question-series. It is the base of the guided discovery teaching. In our Mathematics 6 textbook we placed a separate chapter with the title *“How do we solve tasks?”* It is based on Pólya’s phases and questions. We introduce them shortly:

   I. *What is the question?* We drive the students not only to read, to understand the given questions but to pose questions by themselves.

II. *Data collection, data analysis. Collecting data from tables, graphs.* The pupils need to choose the necessary data to answer the questions. They need to notice the superfluous data too.

III. *Thinking backward. Making a picture, a table.* These are valuable tools to find the relationships between data resp. the solution of the problems.

IV. *Control tools:* substitution into the text, estimation, looking for the last digit (even – odd). Solving tasks not having solutions. Solving tasks having more solutions.

V. *Answering the question.* Let us read the question newly, let us translate the mathematical solution into the context of the task.

One experience: We do not separate the phases *Devising a plan and Carrying out the plan* rigidly. We observed that at solving complex problems many pupils don't see the whole solution plan at the beginning. If they solve a part of the problem without a whole plan, knowing the particular result very often they can continue the solution. Very often it hinders the pupils if we demand them to have a whole solution plan at the beginning soon.
Another experience is that the students should draw figures, sequences at the analysing of the problem. We don't think that the pupils can solve equations on symbolic level before age 12. In the earlier years we prefer the concrete, visual solutions, using for example segments. It will be advantageous in the later years at solving word problems. (Analysis phase: understanding of the problem, noticing relationships)
Unfortunately some teachers tend to go on the symbolic level very early. One teacher told to his class 5: *"We are soon so clever, that we can produce equations, so we don't need to use segments."* I don't know a broad investigation in this topic but our experiences empower the necessity of the use of concrete, visual representations, as it was stated by Pólya himself too."
Some diploma works empower the statement, that the use of modified Pólya's problem solving questions is an effective tool for the students at solving problems. (PINTÉR, K. 2007)

5. Gordana Stankov made a teaching experiment where the topic was the early algebra (Year 7). While solving word problems she has put on the blackboard the following questions based on Pólya. The pupils received these questions on a small card:

What shall we determine?

How do we denote it?

How do the unknown numbers relate to each other and to the given number data.

Write down the relationships with help of mathematical symbols!

Make a diagram to the problem!

Problems:

- To determine the distance taken by a train in 3 hours, if its velocity 60 km/hour is.
- A shoemaker repairs 60 pairs of shoes weekly. How many pairs of shoes does he repair in 3 weeks?
- How much water flows out from a tap in 3 minutes, if in one minute flows 60 litre water out?
- Dan bought 3 pieces of kiwi for 60 forint each. How much did he pay altogether?

Make diagram to these tasks!

What is your experience considering these tasks?

What is the information behind the data: 3 hours, 3 weeks, 3 minutes, 3 kiwis?

What is the information behind the data: 60 kilometres, 60 pairs of shoes, 60 litres, 60 kiwis?

The aim was that students realize that the whole product can be determined by the multiplication of the time and the product in a unit.

Further questions were posed:

1. *What kind of activities are in the tasks? Who realizes these activities?*
2. *What is the measure unit of the time? Or: What is the unit of the activity?*
3. *What is the duration time of the activity? Or: How many units does the activity contain?*
4. *What is the product of the activity in a time unit? Or: What is the product of the unit of the activity?*
5. *What is the total output of the activity?*

The main aim was that students discover the analogy between problems with quite different contexts. (STANKOV, 2008)

6. The phase *Understanding the problem* whether the students the task vs. its solutions really understood is a hard question, very difficult to control it.

   One example: For the real numbers a, b, c yield: $(a + b + c)\cdot c < 0$. To prove that $b^2 > 4ac$.

   One possible solution is: Let us assume that the first argument (antecedent) is true and the second (consequent) is false. So yield $(a + b + c)\cdot c < 0$ and $b^2 \le 4ac$. If we multiply the first inequality by 4 und add to it the indirect statement and finally subtract 4ac from both sides we get: $b^2 + 4bc + 4c^2 < 0$ so $(b + 2c)^2$, but it is a contradiction, so our original statement is true. This solution is correct, but we may ask ourselves: do our students really understood it?

By my experiences better to handle the following solution: The $b^2 > 4ac$ expression can the students to remember for quadratic function. Let us consider the quadratic function $f(x) = ax^2 + bx + c$! The condition $(a + b + c)\cdot c < 0$ means that $f(1)\cdot f(0) < 0$, also the function takes on the places 0 vs. I values with different signs. It follows that our function has a zero place between 0 and 1 and needs to have another zero place, what means $b^2 - 4ac > 0$ (discriminant is positive). It follows our statement is true. Here was a help the heuristics question: *Do you know a similar problem?*

Based on the Pólya phases and questions I told often to my students: *You shall to build a databank, which contain the learnt concepts, definitions, algorithms, methods and ideas. But you need to have a search program too which compares the elements of a new task with the databank and chooses the relevant date.* This search process contains a lot of Pólya's questions: What is given? What do we need to find? Have you met similar problems? Can you solve a part of the problem? (KATZ, 2009)

Summary

We know that the Pólya's problem solving phases were modified for example by John Mason, Leone Burton, Kay Stacey resp. by Schoenfeld, but they are not known between Hungarian mathematics teachers because of their language difficulties. Summarized the teachers' reactions only one teacher used the original Pólya phases, questions, without any changes. The other interviewed teachers applied its modified versions, to mention that they considered the Pólya questions as base.

Teaching to think

Reading the interviews and hearing the teachers seems to us quite clear that one of the most important aim in Hungarian mathematics education is the developing the students' thinking abilities. They did not mention Pólya explicit but I am sure that this tradition comes from Pólya. He said: "To develop the thinking abilities, skills is much more important than the simple material knowledge, although the developing of the thinking skills may built only on solid material knowledge. Today it is a requirement, that the students must have individuality, originality, creativeness." Interesting is that Pólya equates thinking with problem solving. "A *Problem may exist without thinking, tough the formulation of it is a result of the thinking, but* *thinking is unimaginable without problems. What I am thinking about, if I do not have a problem to solve it." (PÓLYA, 1971)*

1. "I think that in Hungarian Mathematics Education the developing of the thinking abilities has a great tradition. Beside that the discovering the relationships and some extensions in the direction of higher mathematics are characteristics." (RÁBAI, 2007)

2. "If we don't teach to think, the mathematics teaching loses its true sense. I think if we have difficulties at realizing the curriculum, let us decrease the teaching material. Important is that the students shall learn to think on the material we are teaching. For example to solve a trigonometric equation has not a true value in mathematics. It is important that the student solving the problem mines ideas from his (her) brain (memory) and start to learn to think logically. Who is able to think logically can profit from this ability on the juristic profession, in the factories and everywhere.
   In mathematics teaching the most important thing is the developing of the purposeful thinking. We need to suborder everything to it. The mathematical notation makes it easier to express our ideas, opinions, but the first is always the thought, the idea, the invention." (CZAPÁRY, 2007)

3. "If we want to achieve that a student can solve a demanding task, he (she) needs to have a firm, practiced, application ready and with relationships powered knowledge. Such knowledge he (she) can recall easily and can apply it in different problem situations. To solve a demanding task needs to apply not only an algorithm, a rule, respectively a simple recall and application of a piece of knowledge. The tasks are not demanding if we can solve them in one or two steps based on a stereotype or on a formula. From the point of view of developing of the students' thinking abilities solving such tasks have not to much benefit although practicing such skills are important too. The essence of a demanding task is to find the usually more basic knowledge necessary for the solution, its recall and application, planning the steps of the problem solving process, finding the different solutions and giving the necessary arguments.

   Example
   In a trapezium the length of the base is 10 cm, the length of its altitude is 4 cm. One of its legs builds with the base an angle 60°, the length of the other leg is 6 cm. What is the area of the trapezium?

This problem has two solutions, but the students often are satisfied with one solution." (TATÁR, 2007)

The author's comment

For this teachers problem solving is the same as thinking, as we have seen at Pólya.

To make the situation more complicate the Hungarian mathematics curriculum contains a chapter in grades 9, 10, 11, 12 *"Thinking methods."* In details: Definition of informal, concrete, visual notions. Number sets. Operation between sets. Combinatorial operations, tasks. "Then and only then" type statements. Theorems and their inverse. Indirect proofs. Pigeon principle. Differentiation between everyday and mathematical thinking. Combinatorics. Simple graphs, solving problems with help of them. Developing of deductive thinking. Logical operations. Mathematical induction. Systematization of proof methods.

Because these questions are separated from other chapters, a lot of teachers consider them as separate topics to be handled, though one should develop them continuously. Because the Hungarian mathematics education is problem centred, of course the principles, operations are embedded into different, complex tasks.

But the question is not so simple. If we look at the PISA competencies we may find separate the above mentioned abilities, competences (NISS, 2003).

Mathematical thinking skills

Mastering mathematical modes of thought:

- Awareness of the types of questions that characterise mathematics,
- Ability to pose such questions,
- Insight into the types of answers that can be expected

Problem handling competence

Being able to formulate and solve mathematical problems, i.e.

- Put forward (detect, formulate, delimitate and define) different kinds of mathematical problems, pure and applied, open and closed.
- Solve mathematical problems, if already formulated, whether posed by oneself or by others, and if necessary or desirable, in different ways.

The first competence does not give explicit information about mathematical thinking. In the J. Mason, L. Burton and K. Stacey problem solving book *Thinking mathematically* we may find typical mathematical activities, questions: specializing, generalizing, conjecturing, discovering patterns, seeking structural links, justifying, extreme values, conditions.

In question forms:

How many ways? What is the most/least? What is the underlying structure?
Will the same technique work more generally? Why is it like that?
Why is it not like that other situation? Why does that happen? What patterns are there here?
Where do these numbers come from? What happens next? Can I predict what will happen in general? What is this situation a special case of? What is going on here?

We can not give a final answer for this question, just have signed the problem. What seems as true is that between thinking and problem solving there is a strong positive correlation.

About teaching methods

Pólya states: "If somebody wants to learn to swim, he must go into the water, it is not enough to hear lectures about swimming. Making movement trials in the water he can discover the best ones by himself. Let us leave the students to go through the painful ways of the solutions of the problems. He should feel the "HEUREKA" experience as often as possible." (PÓLYA, 1971)

A lot of Hungarian mathematics teachers share this idea. Every year a lot of task collections are published with the aim to give the teachers and students nice problems, to deal with them. The question is not simple, a unique meaning is that does not exist a general method for all.

Here are some teachers' opinions:

1. "In our textbook series *at the start of each part there are solved (model) tasks with the aim the students can find analogies at solving a lot of similar other tasks*." (PINTÉR, K. 2007)

2. "I think there are no such methods which are effective for every class, for every student group. The most important task for a teacher is to find the relevant tempo, the abstraction level which prompt effort; our teaching will be really developmental only in this case.

   By my experiences to acquire a solution method or idea on an effective level, the students need to meet them at least three times, which should be separated in time:

   I. The method or idea is shown, explained or deduced by guided discovery.

   II. It should be trained, recalled by repetition.

   III. The idea or method is embedded into a more complex context, where other methods or ideas shall be applied too.

   In 3 – 4 – 5 weekly lessons there is no time to discover everything what is needed to be successful in future studies or competitions. I try to find a balance between the demonstrator role of the teacher and the guided discovery.

I use often a metaphor as I usually tell my students they should build in their mind a database which contains the learnt concepts, theorems, procedures, ideas. Additionally they must have an effective search-program, which compare the data of a new problem with the elements of the database." (KATZ, 2009)

3. "Due to the involvement in the PDTR (Professional Development of Teacher Researchers) programme *I am trying out new teaching methods, new type of examples, exercises. I reflect a lot more and more often on: what happened in the lesson, why it happened; what was good; what went wrong; what I should change next time to avoid it, etc. Although I have been educated in a very teacher-centred, frontal way as it used be the tradition in Hungary, I try to shift the focus of attention to my students by making them working* in pairs, giving them such problems which can be solved without the teacher. I try to reduce the amount of collective discussion because I used to overuse this technique. I can only hope that this way I am giving them enough space in the mathematics lessons. As far as the content of the lessons is concerned, I dare to give more challenging problems to my students and I have to give them more lifelike problems to meet the requirements of the new final exam. I believe that my long term objectives have changed too, namely that I do not only want to prepare them for the final exam but I have realized that I have to teach them how to think, to be ready to put into practice what they have learnt and to feel the need of continuous learning. I have been working on issues like praising my students more often, not being very critical in connection with their mistakes, allowing them to make mistakes, encouraging them to follow their own ideas, not mine. Moreover, I take their efforts more into consideration than before.
   In brief, I am becoming a more conscious teacher who is more aware of her weaknesses than before. This teacher has realized that she has to change the way she teaches maths as the needs of the students and that of the world have changed, too. So, she has been experimenting with new methodology, new exercises and a new attitude towards teaching mathematics. She is also planning to be more familiar with recent methodology literature because it can help her to achieve her goal." (KOI, 2007)

4. "It is not the case that I teach somebody for something. I am teaching him and after that we must go further. Maybe something fit into his mind, maybe nothing. He turns back to this thing sometimes and somewhere and then suddenly it will be clear for him. Professor Surányi told once: *We never teach that what we teach!* I have followed this opinion. We must go forward and to hope that in the students' minds the problem works further. In the

first moment is enough if he understands the melody of the taught material, has some feeling about it" (HERCZEG, 2007)

5."I have a very simple methodical slogan: *Let us start always with the teaching of simple things!* In my opinion it is very important that the teaching of more complex materials we should start with simple steps. The mathematics is very difficult for a lot of students, because at the teaching they spring some steps, saying that thing are so simple that we do not need to teach it.

Example: Before the proving the inequality between arithmetic and geometric mean we should analyze the inequality a ≥ 0. I think it is obvious for all students that is true for all value of a. After this small task it is more understandable the returning back to this inequality

$$\frac{a+b}{2}-\sqrt{ab}=\frac{a-2\sqrt{ab}+b}{2}=\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}\geq 0.$$

We shall not to teach the solution of the tasks, but through the solutions of the problems the solutions methods!

Task series:

Let us solve the following equations:

$\cos 2x = 0$ $\sin 2x = 0{,}5$ $\sin^2 x = 0{,}25$ $\sin 4x - \cos 2x = 0$ $4\sin^2 2x - 8\sin 2x + 3 = 0$

$\mathrm{tg}x + \mathrm{ctg}x = 8\sin 2x$.

You may see the growing difficulty level of these tasks.

I have a slogan to the teacher's work: *Do not leave to forget the knowledge of students!* What important, essential is, let us hold it awake! Do not leave to forget the essence! The most important tool for it are the solutions of the problems. We need to give such problems, that their solution needs such knowledge, methods which are essential in our opinion." (RÁBAI, 2007)

6."We usually interrupt the class teaching with group or pair work. Sometimes I build the pairs consciously, sometimes spontaneously. The students help each other outside of the school too. The 3-4 members groups usually are arbitrary built, what important in this case too, after finishing the work, one group member present the result of the group.

Sometimes I give the solution of a more difficult problem into the hand of a student, he (she) needs to understand and explain it for the class.

After practice lessons I give often homework, the students need to formulate problems similar to the handled problems.

In my opinion is very important to understand, follow the thinking process of others. Once in a year "student-teachers" correct the test written by their classmates." (KELEMEN? 2009)

7.“After discussing the new material we solved practicing problems. The pupils worked individually, later on helping to each other. The best students finished their work earlier, I controlled them and these students helped to the students who could not manage the solution.
In my opinion it is *very important for the students to talk on the mathematics lessons*. If we want to reach that all of our students can follow the logic of the task, it is necessary to explain it loudly by the students, to control it with his mind that the speaker is right. The listeners can observe their classmate to explain the solution of the problem, whether his expressions, argumentations, thoughts are right. It is important to leave the students to speak. It has a great educational power. Additionally I think that it is very important to give freedom to the students to express their opinions, ideas free, they get time and possibilities to discuss, can give arguments to the solution of the problem." (TATÁR, 2007)

8.“*Each man thinks otherwise*. It is wonderful to see how the thinking of the students starts into some direction. They cannot always find the solution without help, the teacher must continue at the part where the student was blocked. From a teacher can be expected that he (she) must know how to handle the pupils. By my experiences what is missing in our mathematics education is the personal care at most students." (PINTÉR, 2008)

Summary

The teachers cited above are good teachers, they are teaching mathematically gifted students. Most of them teach by the following structure: demonstration, explanation – practice – solving complex problems applying the learnt new methods. The other characteristics are the discussion of the solution before the class and the individual care.

How to develop the problem solving competences of mathematically not highly talented but diligent and clever students?

We mentioned that the Hungarian mathematics teaching is elite oriented. Most of our students are not high ability students, but we must develop them too. In the following part we cite three teachers teaching in average secondary schools.

1. "I am teaching mathematics in a secondary school in a small town in the countryside of Hungary, we meet here very rarely mathematically really talented student, but we do have rather diligent and clever students. They want continue their studies on the universities or colleges. Our main aim is the foundation, the preparation of higher studies. A good inter-

mediate step is the preparation for the higher level mathematics maturity exam. The diligent, clever students rarely have original ideas. They can learn a lot of tricks, methods with the aim to bring them out in the right moment at the problem solving process.

How can I achieve that they apply the adequate method in an adequate moment? I show some possibilities, ideas first on simpler problems, so they can focus on the essence of the method. After that we practice it on some tasks and finally they get gradually more difficult problems in some cases embedded in a text."(KELEMEN, 2009)

2.“I have had an excellent student. He was a customer of the additional problems, tasks too. It was observable on him that *the quantity can lead to a qualitative change*. He followed the teacher’s instructions, advices, ideas, solution models, but very rarely he tried individually to solve problems, to read mathematical books. After his maturity exam he studied at the Technical University of Budapest with excellent results and now he is the director of the Nuclear Institute of Technical University Budapest. In my opinion it is a great responsibility of the teachers to increase the knowledge of such students. It needs more preparation from the teachers. This preparatory time is much more then the explicit time working together with the students.” (BOLDOCZKI, 2009)

3.“The Hungarian common national core curriculum designates the development of intellectual capacity of students and their personality as well as constructive thinking, teaching how to use the analogies as the fundamental task of school. It is accentuated as one of the educational goals of school that “it should give integrated, coherent view of mathematics not only as an accomplished, inflexible and austere system of knowledge, but also as particular human cognitive activity and mental conduct”. Skill, logical thinking, proficiency in solving mathematical problems evolving a need for mathematical argumentation, these are among its general development requirements.

Unfortunately, in reality the above mentioned goals can be difficult to attain. I can also confirm this with my rich teaching experience. Most students can solve only standard tasks. When they face a new problem they are not able to use consciously and mobilize the learnt material (definitions, theorems, axioms, proofs) by themselves. Talented students are able to see the main points after some practicing, they don’t only remember but also apply the main ideas. On the other hand, for average and weak students explicit emphasizing, understanding and acquisition of *“reasoning operations“* would be very important. Besides, in case of a lot of tasks the problem-solving strategy itself is, at least, as important as the result.

I hold the developing course five times after the lessons (90 minutes each). At these lessons I followed the methodological thesis of Schoenfeld suggested in ‘Problem Solving in the Mathematics Curriculum’ (1983), and applied during the teaching practice. According to Schoenfeld the roles of the leading teacher are: demonstrative role, moderator and the teacher, the coach. The *Demonstrative role* means that during solving the problems together with the students the teacher demonstrates the possibilities, effectiveness, pitfalls of the use of different strategies, and the process of problem solving itself. As a *moderator* the teacher must use the students’ proposals, react to their questions, and guide them with proposals, suitable questions. The *coach* role means that the teacher has to show, stress the simplest way and method in case of problems accessible and solvable in more ways with ‘*… try to do it in this way’.*

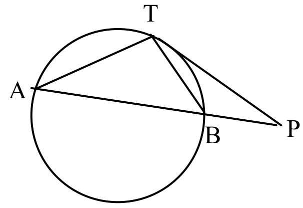
One example: First I was talking about the types of the strategies, the different ways of thinking, the question which they have to ask from time to time when proving a theorem. I made the following board-draft:

Forward method	*Backward method*
Let’s prove the theorem: $\boldsymbol{A \Rightarrow B}$	
A: the condition of the theorem	*Starting point: B statement of the theorem*
B: the statement to be proved	*To B we are looking for an* E_1 *sufficient*
K_1*: the consequences of the interim steps*	*condition from which B derives*
$A \Rightarrow K_1 \Rightarrow K_2 \Rightarrow \dots \Rightarrow K_n \Rightarrow B$	$B \Leftarrow E_1 \Leftarrow E_2 \Leftarrow \dots \Leftarrow E_{n-1} \Leftarrow E_n \Leftarrow A$
Questions:	*Questions:*
What are the conditions of the theorem?	***What does the theorem state?***
What derives from the condition?	*From what does the theorem derive?*
What is our goal?	

I called their attention to the importance of wording and writing down their arguments, references to certain theorems, axioms, the data of the condition during problem solving every time. During this lesson we solved some tasks of the pre-test with the help of the discussed strategies.

One example:

Prove if PT is the tangent of a circle, PB is the secant of this circle, and A is the other point of intersection, then PA·PB = PT^2!



Applying forward method

During problem solving the defined aim is always in the centre of our thinking. Our questions: What is the goal? What are the conditions of the theorem? What derives from the conditions?

Our way of thinking is the following:

K_1: Let's link points A and T, and B and T points (draw the TA and TB segments).

K_2: Let's examine TAP and TPB triangles!

the angle at P is common

ATP ∠ = PBT ∠ (they are perimeter angles on the same arc)

K_3: TAP Δ ~ TPBΔ derives from the previous steps

K_4: From this derives the equality of the corresponding sides, that is: $\frac{BP}{TP} = \frac{TP}{PA}$.

K_5: We reach the statement with applying the characteristic of the proportion.

Applying backward method

The questions that we ask before all the steps applying this problem-solving strategy are the following: What does the theorem state? Where does the given statement derive from?

The process of our thinking is the following:

E_1: To show the statement to be proved, it is enough to prove the next equality: $\frac{PT}{PA} = \frac{PB}{PT}$.

E_2: To prove the equality of the proportion of the two segments, it is enough to find two similar triangles. With the segments we examine the APT and PBT triangles.

E_3: To prove the similarity of PAT and PBT triangles, it is enough to show the equality of two pairs of equivalent angles:

ATP ∠ = PBT ∠ (they are perimeter angles on the same arc)

TPA ∠ = TPB ∠ (they coincident).

The latter statements derive from the conditions of the theorem. We have proved the original statement." (KOZÁRINÉ, 2009)

Summary

I could follow for a long time the mathematics teaching of teachers cited above, their hard work from lesson to lesson. It were quite clear for me that the model of teaching of gifted students does not work for others one. If we want to reach and mathematically develop more and more students, we must go forward step-by-step with help a lot of explanation, demonstration, individual help, using not only symbolic, but visual, concrete representations too. Last but not least very important is for the teachers to have a very good emotional relationship with their students.

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The design of an assessment model, including elements of problem solving, in upper-secondary school mathematics

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Abstract

As considered in research in mathematics education, there might be a significant mismatch between assessment and instruction including elements of problem solving, if the assessment is very traditional. In this article, the design of a model for assessment, including elements of problem solving, and adapted to Finnish frame factors, is discussed.

Keywords

assessment, problem solving, design, upper-secondary school mathematics.

Assessment in mathematics education

In the book "Investigation into Assessment in Mathematics Education", Mogens Niss gives an introduction called Assessment in Mathematics Education and its Effects (Niss, 1993). After stating that assessment in mathematics education is considered to concern the judging of the mathematical capability, performance and achievements of students, he continues by saying that the developments in the field of mathematics education have not been matched by parallel developments in assessment. According to Niss, we face an increasing mismatch and tension between the state of mathematics education and current assessment practices. We also face a widening gap between contemporary mathematics teaching and traditional assessment practices. (Niss, 1993, 1-30)

Niss presents, what is assessed is *predominantly*

a) mathematical facts

b) standard methods and techniques

c) standard applications ,

in an increasing but limited number of cases

d) heuristics and methods of proof

e) problem solving

f) modelling,

and *rarely encountered* are

g) exploration and hypothesis generation .

Niss proposes to place more emphasis on d) – f) and g).

Obviously, this mismatch occurred despite of earlier and contemporary good efforts. For instance, in a pamphlet from the National Council of Teachers of Mathematics called "How to evaluate progress in problem solving", Charles, Lester and O'Daffer (1987) gave several guidelines for implementing an evaluation program including advices like "evaluate students' work on a regular and systematic basis", "match evaluation plan to instructional goals", "observe students' small-group efforts and their written work as an important part of evaluation plans" and "advise students of the evaluation plan and how it works".

Niss' suggestion in 1993 seems to imply that advices like these were not followed to any reasonable extent by teachers. Of course, this fact could also be explained by for instance difficulties to do time-consuming observations in class and difficulties to assess individuals' work in groups.

Assessment as a part of instruction

Some years later, Lorrie Shepard addresses the problem related to assessment in her article "The Role of Assessment in a Learning Culture", in which she emphasizes the kind of assessment that can be used as a part of instruction to support and enhance learning. Like Niss, she also finds the situation unsatisfactory and she gives the statement that instruction is drawn from the emergent paradigm, while testing is held over from the past. Moreover, she urges that classroom assessment must change in two fundamentally important ways: its form and content must be changed to better represent important thinking and problem-solving skills, and the way that assessment is used in classrooms and how it is regarded by teachers and students must change. (Shepard, 2000, 4-14)

Consequently, Shepard finds a need for

a) a *broader range of assessment tools*, including for instance open-ended performance tasks to ensure that students are able to reason critically, to solve complex problems, and to apply their knowledge in real-world contexts

b) a more direct connection between assessment and on-going instruction, including the possibility of (*classroom*) *assessment integrated in instruction* in order to support learning

c) more *formative assessment to enhance learning*, and more assessment of processes and not only assessment of outcomes . (Shepard, 2000, 8)

Shepard suggests seven specific assessment strategies to be effective, if they are part of a more fundamental shift in classroom practices:

S1 Dynamic assessment
S2 Assessment of prior knowledge
S3 The use of feedback
S4 Teaching for transfer
S5 Explicit criteria
S6 Student self-assessment
S7 Evaluation of teaching

Concerning dynamic assessment, Shepard points out that assessment should be moved into the middle of the teaching and learning process instead of being limited to only the end-point of instruction. Thus, assessment can provide possibilities to enhance learning and for instance, test preparation can form a better base for the following steps in teaching.

The great advantage with assessment of prior knowledge is that the students know what they have to learn when (or rehearse before) they are taking part in the next instructional activity. Here we also have the possibility to integrate assessment into instruction.

It is generally thought that feedback to the learner will lead to self-correction and improvement. The motivation for corrections may be a true intention to learn or just the hope for a better mark in the end-of-the-course test. Hopefully, the motivation is the former one, because then, the impact is likely to be of greater value.

Shepard stresses the close relationship between truly understanding a concept and being able to transfer knowledge and use it in new situations. Thus, it should be a desirable goal for the teacher to have the students not only to master the classroom routines but also the underlying concepts. Accordingly, we should not agree to a contract with our students saying that the only fair test is one with familiar and well-rehearsed problems. Instead, we should include, for instance, real problems and project work with a starting point in the real world. Rich tasks in this respect (Burman, 2009, 53-59) could enhance students' transfer of school mathematics to their own world outside school.

The strategy of explicit criteria implies that students must have a clear understanding of the criteria by which their work will be assessed. This strategy is different from the previous four strategies in the way that it includes components of teachers' general behaviour in the contact with students but the significance in mathematics instruction is also clear.
Students' self-assessment and an increasing responsibility for their own learning is hopefully

the consequence of an assessment included in the instruction. The students are invited to choose when they will learn: "as soon as possible", just before an end-of-course test or somewhere in between. Again, hopefully, the students should think "the sooner the better", in order to easier follow the next steps in the course and consequently, have better chances to pass the course (with a better mark).

Finally, according to Shepard, classroom assessment should also be used to improve teaching practices. In comparison to the strategies above and the focus of this article, a natural component in this respect is to make the inclusion of more elements of problem solving visible to the students.

The aim of the article

The aim of this article is to design and present a model for assessment, including (more) elements of problem solving. More precisely, the aim is, that problem solving, in a wide sense, to a greater extent should be included in course content as well as in assessment. Moreover, the model should be adapted to Finnish frame factors, i.e. it should be designed for use in a Finnish upper-secondary school with the Finnish system of periods and courses in mathematics.

Finnish constraints

In Finland, the school year is 190 days long, and simply divided by five, this means 38 weeks. In Finnish upper-secondary schools, the school year is often divided into five (or six) periods of about seven (six) effective weeks. The content is put into courses with the same length, about 18 lessons of 75 minutes (or 30 lessons of 45 minutes). A short course in mathematics consists of six compulsory courses and an extended course in mathematics of ten compulsory courses + 1-5 extra courses.

In Finnish upper-secondary schools the students receive two final marks, one from the teacher(s) and one from the Matriculation Examination (ME). The ME, where the students should choose 10 tasks from 15 tasks and solve them in 6 hours, has a great influence on assessment and consequently also on the instruction. In many schools, a course in mathematics is assessed by the use of a final exam of the same type as used in ME.

The model for assessment

Tests during the course

The design of a model for assessment needs a theoretical base, and in this case, Shepard's assessment strategies, S1 – S7, serve as the starting point. The dynamic assessment mentioned in S1 implies that an end-of-course type of assessment in unsatisfactory. There is a need for some kind of assessment in the middle of the course and preferably, the assessment should occur repeatedly. If the final test consists of about seven tasks of the type, which is common in the ME mentioned before, another seven or eight tasks could be used during the course. Eight is easily divided by two and thus, four tests with two tasks in each could form a series of *minitests*.

Four minitests also give the students feedback about their current knowledge four times during the course. If there are lacks in knowledge and skills, the students realize what areas they have to improve. The need to learn mathematics or just the desire for a better mark in the course may be the sufficient motivation and thus, minitests can also serve as the answer to the requests in S2 and S3.

Presumably, unfamiliar tasks, real problems and projects might improve the students' competence to make transfers (S4) from school mathematics to the world outside school. A minitest could include tasks which are different from those in the final tests. For instance, solving a real-world problem or just making a description of a method to solve a given problem could be the means of testing the understanding of a concept or a process. Moreover, it could be the means of drawing the focus to understanding and not only to finding a proper answer to a given question.

No matter how assessment is performed, it should be clear of all reasons that the students should receive information concerning the criteria for their assessment. Of course, there are several possibilities for us to fulfill the demand in S5. If the students can collect useful points, they tend to do it and thus, it is important for the teacher to give points for the tasks in the minitests, as well as for the tasks in the final test.

The minitests offer the students a possibility to take responsibility for their own learning by increasing the element of self-assessment, which is highlighted in S6. It is a clear advantage for the students, if they perform well already in a minitest. Of course, they receive points for good answers. But furthermore, as new steps in mathematics often are based on previous knowledge, it is easier to follow the on-going instruction, if the knowledge and skills have been stabilized by the preparation for a minitest. Thus, the minitests could have two kinds of effects, a direct one, the points, and an indirect one, more effective learning in the course.

Finally, there are also relations between minitests and the remaining strategy, S7. The teacher can observe misunderstandings and lacks in the students' skills. From the teacher's point of view, the minitests also provide a possibility to make important things visible to the students. A good example is to stress more elements of problem solving and not only tasks similar to those of ME.

Project work

There is a demand for projects in mathematics. Projects, with a starting point in real-life problems or areas of students' interest, are also well in line with the transfer strategy S4. In the Finnish system, the courses have very often an extensive content, and it may not be possible to implement project works in every course. All areas of mathematics do not offer good connection between the on-going instruction and a modeling project. On the other hand, there are courses in statistics and courses including linear and exponential growth, which can be regarded excellent in this respect.

As with minitests, S5 demands project users to tell the students how they are going to be assessed. Of course, in comparison to other sources of information, project works could be weighed in different ways, and perhaps the weight could be different in different courses.

Elements of problem solving

As stated earlier, including more elements of problem solving and especially real problems and projects might improve the students' competences to make transfers from school mathematics to the world outside school. Application tasks in a Finnish ME may quite often be considered to possess only a simulated context and fail to be truly authentic (Palm and Burman, 2004, 1-33). A task is more valuable if it is authentic and the connection to the real world is important, as the students' transfer of school mathematics to their own world outside school is desirable (Björkqvist, 2001, 116-118).

Reliable assessment

Minitests, as supplement to an end-of-course test, increase the information about the students' knowledge and skills. Moreover, minitests can be regarded as a kind of dynamic assessment (S1), and they collect evidence of knowledge and skills from several different occasions. Minitests also give information for assessment purpose more often, which is in line with the strategy of prior knowledge S2.

Consequently, minitests give useful feedback (S3) to the students, for instance hints of what to rehearse. Thus, the students can be better prepared for future tests, i.e. minitests also enhance self-assessment (S6). As mentioned before, there is a so called indirect effect, which means that the use of minitests have a greater effect on the final result than the points from the minitests themselves.

Tasks that enhance understanding and transfer (S4) will simultaneously provide possibilities to widen the base of assessment and thus, make it more reliable. Of course, there is the problem of choosing the right weights to all different sources of information (S5), in order to get a fair final result for each student. However, there are no general answers to that problem. It has to be solved differently in different situations.

As S7 has connections to the assessment as a whole, we notice, that the strategies S1 – S7 are playing an important role as the model for assessment using minitests is designed.

The model and some final remarks

There are four components in the model:

M1 *Minitests*

M2 *Modeling projects*

M3 *More elements of problem solving*

M4 *More reliable assessment*

As a result of the Finnish constraints, there is the intention to avoid a big shift in instruction, as ME is so important in Finland. Instead, the model provides opportunities to proceed with small steps in the right direction. The recommendation could be to use four minitests in each course, to use a (rather small) project in some of the courses and to include (more) elements of problem solving in the minitests and in the instruction as a whole. The assessment model aims at improving learning by improving assessment and the instruction, into which the elements of assessment are integrated.

The design method used in this article could be described as follows. At first, the general situation is recognized using Niss' proposal. Then, Shepard's seven strategies are used to describe an improved situation to aim at. Finally, the assessment model is designed with a constant attention to the context, i.e. the Finnish constraints, because the model is supposed to be used in Finland.

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About the conditions of elementary school teachers' and secondary school teachers' training in Hungary on the basis of comparison of mathematical problem-solving skills

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Abstract

The introduction of the 2-phased Bologna-education must have solved several problems, but could very well have brought up even more. One of the most important problems is that only a very limited number of BSC students of sciences choose to be majors in the faculty of teaching. To solve the shortage of necesseray teachers the Hungarian Ministry of Education introduced a new form, since last year the teachers of Arts, having been educated for a long time in huge numbers, have the chance to graduate and acquire a MSC degree in teaching mathematics within a year after the completion of the introductory mathematical subjects worth 50 credits. At the same time in case of the students of teachers' training colleges who have completed their studies specializing in Mathematics for 4 years only very few credits are to be considered. If they are about to get an MSC degree, they should start their studies practically at zero. This is a relevant issue, since their number is also quite high. We think that this system is not good and not rightous. This the reason why we have made a survey of the graduating university students, majors in teaching mathematics and of the graduating students of teacher training college, regarding their basic preparational level focused on mathematics teaching at schools. On the basis of this we are trying to draw the necessary conclusions. In the frame of our experiment we gained results of a wider scope than our original aims, and gave some additional directions concerning the coming researches.

Introduction

In 2008 joining the survey of the CIMT (Centre for Innovation in Mathematics Teaching) we carried out a survey related to the mathematical problem-solving skills of the students graduating from Debrecen University, major in teaching Mathematics. The outcome of the survey was fascinating, furthermore, we decided that it would be even more beneficial to extend the original survey. Before presenting the essence of this decision, it is worth remarking here that with the introduction of the Bologna procedure, and of the two-phased teacher training, the latter one has dramatically lost its popularity. There are majors now, especially in the field of teaching sciences, where there can hardly any interest be observed all over the country. The Ministry of Education came to the conclusion that changing of the majors should be facilitated and accelerated. In this way, let us say that a colleague of ours, a teacher of Hungarian and English, having graduated from our university decades ago, decides to complete the Mathematics or Informatics teacher major, after the completion of 50 credits he/she can do so within a year, at the cost of one weekend instruction a month. Also, if he/ she still feels like studying, within the originally planned 5 years he /she might even finish the major in Biology or History besides Mathematics that was mentioned in our example. It does not take a fortuneteller to see that this procedure, highly –disputed by university employees, is going to lead to a further deterioriation of the level of education at schools.

The model established in Hungary based on the Bologna system has – to a certain measure – its deficiencies. Let us consider the case of the students who obtains their certificate at a teacher training college entitled to instruct would-be teachers of classes 1-4. Let us suppose that they graduate in the Mathematics field of education. During their 4-year-long-studies the selected people learn the following subjects in Mathematics: The basics of Mathematics (1st semester), Basic Algebra (2nd semester), Geometry (3rd and 4th semester), Combinatorics, Counting and Probabilities (5th semester), Didactics of Mathematics (5th and 6th semester) ,Statistics (6th semester), Functions (6th semester), Basic Mathematics (6th, 7th semester) and a compulsorily selected subject of Mathematics (8th semester). Students who completed that study, receive 9 credits after the courses belonging to the basic training, 33 credits may be obtained in the education area training. So, these students accomplish on mathematics nearly 20% of the credit amount of the full training during their 4 year teacher training studies. Considering their exam obligation: 5 final examinations are made, 11 practical marks are to be collected, and in the 7th semester they are to take a complex college-examination on mathematics. These completed subjects, according to the legal regulations are not worth anything when compared to the university major in Hungarian or English (In which majors, naturally enough, not a single subject of Mathematics should be completed.)

When somebody in Hungary decides to be an educator, he/she may select an academic specialization (what he/she will teach at a school then) or class (that is, wishes to deal with children of a certain kind of age). It seems that the first factor enjoys an absolute priority for the time being, the role of the other viewpoint is currently insignificant. This is indicated by the fact, that by gaining 110 credits of a totally other academic specialization (with 50 of primers and with 60 of vocational trends) it is possible to obtain a new university-level teacher's degree. Yet for a student of a college-level teacher training belonging to the same specialization, on the other hand, it is necessary to accomplish 260 (110 of BSC and 150 of

MSC) credits. (The college-level teacher training lasts for 4 years in opposition to the old, university training, that lasted for 5 years when it included the additional training.)

In the transformed teacher training the primary school techer education has namely uniquelly preserved its uniform model, to be allowed to say its ”undivided” training of 4 years that is 8 semesters. It has got a well-defined training and output requirement system, following the educational politics, that makes the role of the graduating students in the labour market of the educator society unambiguous. This means a lot if we take into consideration the so called status insecurity of the freshly graduated firstever BA or BSC graduates in Hungary.

In our opinion it would be worth comparing the mathematical knowledge of the graduating students of the teachers' training colleges and those of the students graduating from university, majoring in Mathematics so as to draw the consequences. We carried out this by means of the CIMT-questionnnaire, which assesses the problem-solving skills needed for teaching at secondary schools. (CIMT gave its permission for the questionnaire to be used for our own purposes)

The questionnaire consists of two parts. Part A includes 15 tasks which can be solved without higher Mathematics. Originally part B consisted of 16 parts. We chose 11 tasks out of them for whose solution the material of the teachers trainig college is sufficient. 52 university students and 14 college students were involved in the survey. (Actually all the 50 graduating students of the Debrecen Teachers Training College were taking part in the survey, but only 14 of them were provided an advanced level of Mathematical training.)

The outcome of the survey

The most fascinating fact is that there were several tasks at which the students of the teachers training college were better.These are as follows: A3, A7, A8, A12b, A14, A15 and B4, B7, B8. Let us have a more thorough look at them.

The task A3 is a simple calculation, where the value of harmonic mean should be determined with substitution. A7 is a task that can be solved with a simple logical conclusion. A8 is a calculation with normal forms. All the three parts of A 12 required decisions of the true statement related to squares, whereas in case of A15 the existence of a triangle had to be proven.A14 was also aritmetical task, calculation of percentage. As for B tasks, B4 required the determination of the set of values of a function, whose formula was given. As the solution of B7 the sum of a geometrical progression had to be given, and for that of B8 the sum of a mathematical progression.

It can be easily observed that these tasks do not present a problem at the given level, that is to say students did not have to carry out mental activities for the solution whose mode were not known at the emerging of the task. Noteworthy that these routine tasks were better solved by those students who graduated from the teachers training college. It indicates that tasks requiring lower level of logical skills are managed with higher reliability by students with lower level of education.

Let us have a look now at those tasks which need productive thinking of problem-solving during which new relations are concluded on the basis of the available data. Would that be true that higher level of education presents an advantage at the solution of these tasks?

Quite significant difference (25 %) manifests itself in favour of the university students in case of tasks A4, A6, A10, A12a and A13a and B1, B5, B9. In what follows we are going to have a look at them.

Compared to the previous one, A4 requires a much more complicated solution. On calculating the third side of a triangle, the type of the triangle should be determined and on the basis of it a question related to the concerned angle should be answered. A6 is an open problem, which does not have an obviously determinable solution. For A10 the awareness of such a theoretical material is need that is not often used. The material which presents itself in A12a, though can be figured out in a logical way, is not included in the secondary schools' material. A13a is a task which suggests the wrong answer with its question (and as such is uncommon in the periods following the primary school education).

In case of B1 there are two ways available as starting points. The one that seems to be plausible, turns out to be the one leading to a more complicated solution. B5 is a task, a similar to whom does not occur in teaching. Therefore, while solving it, one could not have started with a previously-known formula. In case of B9 we are also faced with a irregular question. In this the only thing to determine is not the roots but the number of them.

Having seen the set of tasks we have concluded that the students of the teachers training college can manage the tasks solvable on the basis of well-known algorithms better, with higher precision. The university students, possessing greater special knowledge, were a bit more careless, whereas more creative, initiative at the same time. György Pólya has observed the same, saying that the thorough, factual knowledge is the prerequisite of the intuitive way of thinking. Students not possessing this, generally lack in intuitive ideas.

We have realized that the conclusions of the results of the experiment are rather complicated. There are tasks at which one of the groups did better, whereas in case of other tasks the other group. However, the university students scored better in several tasks, and the differences were significant in their favour. All in all the average of the students of the teachers training college was: 47.43%, whereas in case of the university students: 70.97%. From the point of view of the problem-solving not the size of the difference was of interest, but the fact that we had the chance to realize that the two groups solve the problems with different strategies. Bearing that in mind, can we arrange the further tasks in their education. As an author of coursebooks, and as an educational specialist one can draw other consequences as well. In Hungary in public education, there is only one type of a book being used in one school most probably to make it easier for teachers to prepare. Therefore the very same book should be used by those students learning in the human faculty in a secondary school as by those learning in the faculties of sciences. This, as we have witnessed, is a very bad solution. On the other hand, we, authors of schoolbooks, should realize it even better than earlier that schoolbooks for less motivated students can not be written in such a way that more difficult parts are discarded from those books that were designed for students with better skills.

The method as to how this survey should be continued can be determined quite easily. In order to get more reliable results, it is worth extending the survey to involve groups of higher numbers. Furthermore , if we intended to indicate how hard the recently introduced system is on the students of the teachers training college, we should carry out the survey involving majors of different subjects, for example teachers of Hungarian and English, mentioned earlier in the introduction. Presumably their results would be even worse than that of the college students.

The questionnaire

Part A

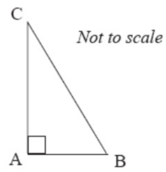
1. Simplify as far as possible $\frac{\sqrt{147}}{\sqrt{3}}$

2. Calculate $(125)^{\frac{1}{3}}$

3. Let $a = 2$, $b = -1$. Calculate the value of H when $\frac{1}{H} = \frac{1}{a} + \frac{1}{b}$

4. Triangle *ABC* is a right angled triangle. *BC* = 12 cm and AC = $6\sqrt{3}$ cm.

What is the size of angle *ABC*?



5. A ball is dropped from a height of 12 metres. It bounces on the ground and reaches ¾ of the original height. It continues to bounce in this way, each time rising to ¾ of the previous height. What height does the ball reach after three bounces?

Give your answer as a fraction.

6. Factorise $x^2 - 7x + 12$

7. Tom, Dick and Harry have a sum £ 575 to be shared among them. They agree to divide it so that Tom gets £ 19 more than Dick, and Dick gets £ 17 more than Harry. How much does Tom get?

8. Calculate $\left(4.2\times10^{-3}\right)\div\left(0.7\times10^{2}\right)$

Give your answer as a decimal.

9. A bag contains 5 red counters, 4 blue counters and 3 white counters. Counters are taken out in succession and are **not** replaced. What is the probability of obtaining two red counters for your first two choices?

10. The length of each side of a cube is multiplied by 3. By what amount is the surface area of the cube multiplied?

11. There is a large number of 5 different kinds of sweets in a bag. What is the least number you must take from the bag (with your eyes closed) to make sure that you get at least 3 of the same kind?

12. Mark each of the following statements as

A: always true   S: sometimes true   N: never true

a) Quadrilaterals tessellate.

b) A square is a rectangle.

c) A trapezium has at least one line of symmetry.

13. Are these statements true or false? Write T or F in the boxes.

a) If the result of squaring a numer is 49, the original number must be 7.

b) All prime numbers are odd numbers.

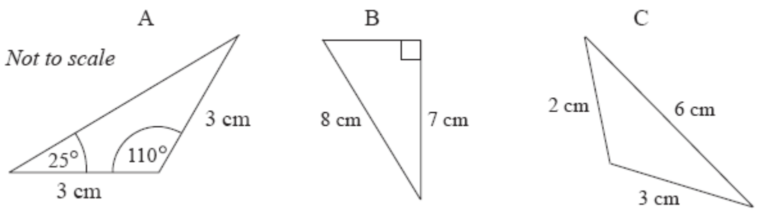
c) The lengths of the sides of a triangle are a, b and c. If $a^2 + b^2 = c^2$, then the triangle contains a right angle.

d) Multiplying a positive number by another positive number always results in a product which is greater than the original number.

14. The price of a television set was increased by 20%. In a sale, its new price was reduced by 20%. How does this price compare with the original price? Is it

A: the same B: less C: more?

15. Which of these triangles can actually be constructed?



Part B

1. If $x^2 + 6x - 3 = (x + a)^2 + b$, calculate the values of *a* and *b*.

2. Determine the number of real solutions of this quadratic equation.

$$2x^2 - 6x + 9 = 0$$

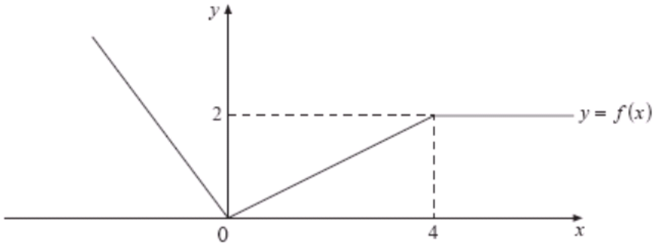
3. If $\frac{6}{2 - \sqrt{3}} = p\left(2 + \sqrt{3}\right)$, determine the value of *p*.

4. What is the range of the function $f(x) = x^4 + 1$? Choose from A, B, C, D or E.

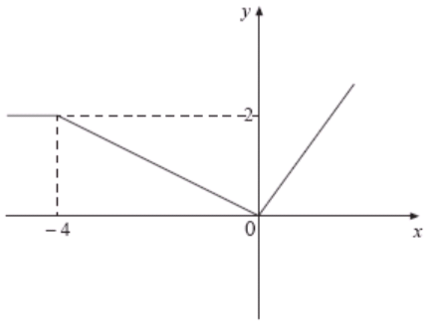
A: $f(x) > 1$ B: $f(x) \geq 0$ C: $f(x) > 0$

D: $f(x) \geq 1$ E: $f(x) > 2$

5. The graph of $y = f(x)$ is shown below.



The graph is translated to give the graph below.



Which of these expressions is the equation of the new graph?

A: $f(x) - 1$ B: $f(-x)$ C: $f(x) + 2$

D: $f(x - 2)$ E: $-f(x)$ C: $f(x + 2) - 1$

6. The equations of two lines are given below.

$$y + 3x - 6 = 0 \quad \text{and} \quad y - 7x + 5 = 0$$

Which of the statements below is *true*?

A: The two lines are parallel.

B: The two lines are perpendicular.

C: The two lines both have *positive* gradients, but are *not* parallel.

D: The two lines both have *negative* gradients, but are *not* parallel.

E: *None* of the above is true.

7. An infinite geometric series begins

$$5 + 2.5 + 1.25 + 0.625 + \ldots$$

Is the sum of this series finite? (Write Yes or No)

If Yes, what is the sum of the series?

8. An arithmetic series has 20 terms. The first term is 2 and the last term is 44. Calculate the sum of the series.

9. How many solutions does the equation below have in the interval $0° \le \theta \le 360°$?

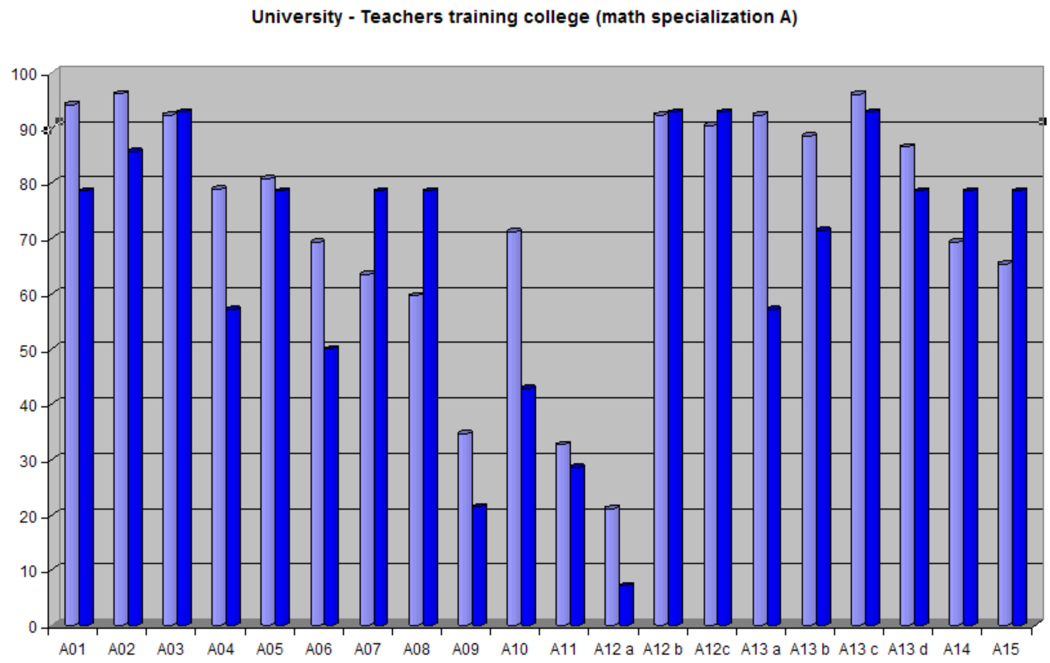
$$8 = 2 + 5 \sin 3\theta$$

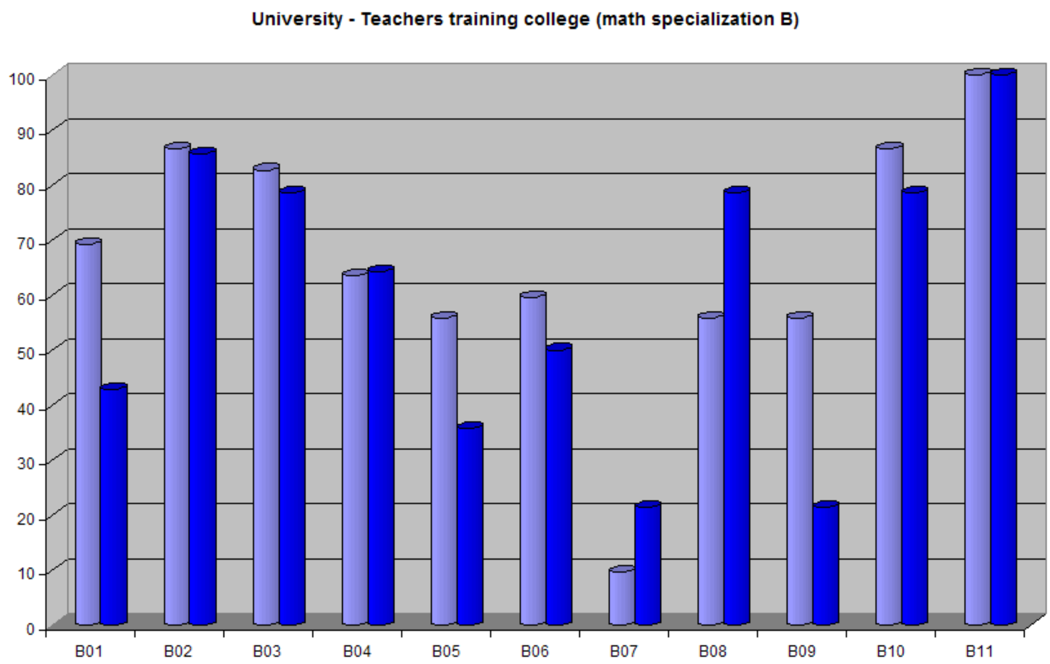
10. Simplify $\log_3 \left(3^4\right)$

11. Evaluate 3^0

The results

Here we can see the detailed results of the experiment.





Conclusions

The main reason of the introduction of the Bologna-based training was that this kind of learning form provides a passage of a high level for students in their and in all EU countries. As we saw from the article, this goal was not achieved in full. In addition, there arose a previously unknown huge problem. It turned out that in the Bologna-system the number of master students in mathematics drastically decreased. While only 40 candidates in the 2009/2010 academic year in Hungary have signed some MSC teacher training course of the field of mathematics, the number of teachers who reach retirement age is about at least ten times more.

In such circumstances every opportunity should be grasped to increase the number of students in science and mathematics teacher. And it would be possible to do so. The students of teacher training college would like going to university because of that, as the knowledge of the students of the mathematical special training in colleges does not differ significantly from the one of university students, as our survey pointed it out. Of course, there are also differences in knowledge and by solving problems. But they are not at a level which could not be treated as a bit of goodwill.

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BIG-small

Sándor Dobos

Abstract

We start comparing things already before school or kindergarten. In this paper we present a chain of problems which follow the long process of building the concept of order. We get an illustration of the step-by-step method, we see how the problems get harder and more complex. In spite of the fact that every problem is around the BIG-small relation, we will visit different fields of mathematics, such as combinatorics, algebra, graphs and geometry.

Keywords

relation, order, transitivity, concept-building, chain of problems, directed graph, problem solving, school mathematical practices

Introduction

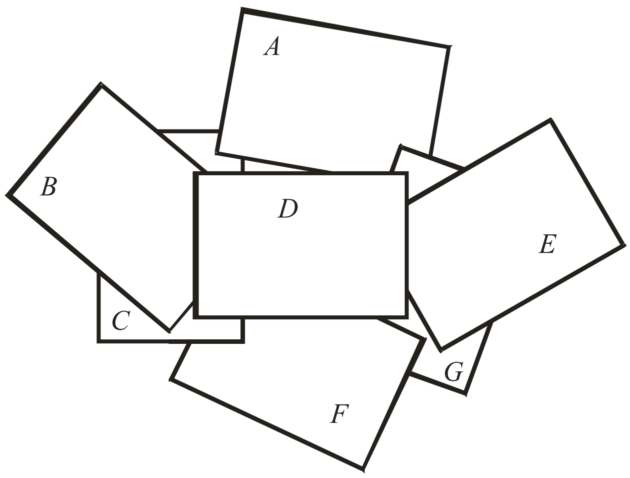
The simple question that among two numbers which one is bigger can be asked even at an early age. Probably this is the first time for a child to meet a mathematical relation, the words big-small, many-few start to mean more and more for him or her. Later in primary school fractions, in secondary school different values of functions are compared. A next step is when it turns out that the size of infinite sets can be different, or that no "normal" ordering relation can be established among the complex numbers. In this paper we cannot follow this complete arc, but try to show the beauty of some parts of it. There are various aspects of research about problem solving, we will focus on the problems, the building of them, the chain of them. In fact, problem solving can be learned only by solving problems (Engel, 1998).

Having teaching experience from the age group between 11 and 20 I have noticed the importance of systematic building. Building a concept is a long process, we need to return to it at different stages of understanding and various fields of mathematics. We try to follow a possible path along the teaching of the ordering relation. There will be a demonstration of the development of problems I have used in class. In the warm up section the first problem gives motivation, the next two follow the chain with a colourful variety of different ideas. The problem chain will lead us to combinatorics first. In the algebra section Problem 6 gives a step-by-step building of questions about comparing fractions according to the level of difficulty. A great chapter of algebra is about inequalities, we just pick one thing from here. Problem 7 is a possible introduction for the rearrangement theorem. The graphs and geometry part take us back to the ideas of the warm up but in a higher level of understanding. The last problem I have heard from János Pach (Columbia University) and I was very happy to find the special figure for n=5.

With interesting questions we may enjoy even the introductory level a lot. Changing some details of the initial exercise may lead us to hard problems. In this paper we plan to have a bird's-eye view of a part of the arc of teaching the "BIG-small" relation from simple to hard questions. The method we follow is to look at problems, their solutions and then we add some didactical or mathematical remarks.

Warm up

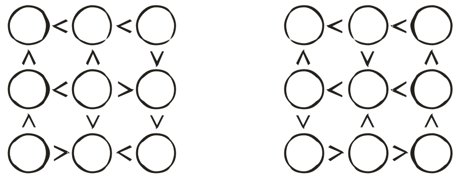
Problem 1. Sherlock Holmes worked in the office of the police station on a delicate issue. The question was whether the policemen were all reliable or not. Holmes was sitting at his table looking at the notes which were on separate sheets of paper and one by one threw them behind him. At a certain moment there was some noise and he turned back. The figure shows how the papers were on the floor. Sherlock Holmes said that in the same room he worked somebody was suspicious. Why?



Solution: Let us denote the papers by letters. We may determine the order of the papers, which has arrived earlier, which later. Clearly the last one must be *D*. Let us remove this paper and now we may remove also *B* and *E*. The arrangement of the remaining four pages helped Holmes. We may notice that *A* is on *C*, so *C* was thrown earlier than *A*. The same way *C* is on *F*, *F* is on *G* and *G* is on *A*, but it is impossible to have such a circle.

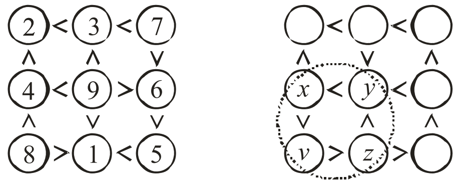
Remark: This problem is suitable for kids even during the first years of primary school. We may also ask it later. The charm of the question is that it does not seem to be like an official mathematical problem. But it draws attention to the transitive property of a relation. However I would not emphasize, or even call it by name. To know the name of the proper terminology does not help to understand the concept of transitivity. Instead we will use this type of reasoning several times and in the meantime the pupils themselves will discover the importance of this property, why it is really worth to give it a name.

Problem 2. Put the numbers 1,2,...,9 in the circles according to the given relations.



2.1. 2.2.

Solution: There are many solutions of 2.1. One of them is given below. It is not possible to solve 2.2. One reason is that there is no place for number 9 since each circle has got a "neighbour " which is greater. Another reason is that there is a circle in it, denoting the numbers by variables we get the contradiction $x<y<z<v<x$.



2.1. 2.2.

Remark: The two parts of this problem seem to be similar but they are quite different. The first one is easy, gives quickly the feeling of success to the pupils. It would be natural to ask, how many solutions there are for 2.1, but this question is much harder so I usually postpone it, but we will see a baby version of it in problem 4. Although neither 2.2 is hard, it blocks more pupils since the solution is not that easy to grab, it is not so concrete. According to my experience both of the above given reasonings are natural, the students find both of them. We used the transitive property again, like in problem 1.

Problem 3. (3.1.) Put the numbers 1,2, ... ,9 in the circles according to the given relations.

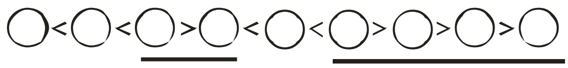
(3.2.) At problem 2 we have seen two figures, the first one had a solution, the second had none. Problem 3.1 has got a solution. Keeping the circles along a line could we put the relation signs between them so that there were no solutions?

Solution: 3.1 is just a warm up to 3.2, it is even easier than problem 2.1. A possible solution is:

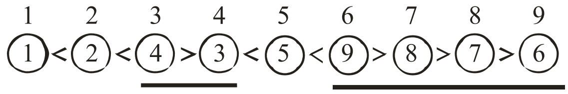
We now show that for any arrangement of the relation signs there is a solution. Let us see four different reasonings.

(i) "Induction-type." Put the nine numbers into a basket. Take the leftmost circle and the next sign. If it is <, then take out the smallest number from the basket, and put it in the leftmost circle. If it is >, then take the largest number from the basket. Continue this algorithm from left to right, take the next circle and the smallest or largest number among the remaining ones. We solved problem 3.1 with this method.

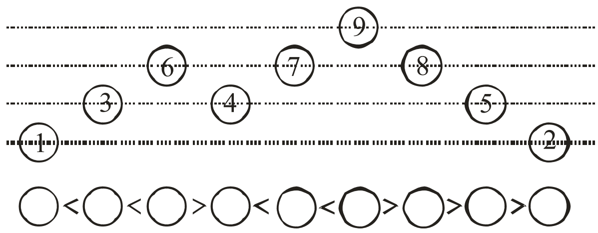
(ii) "Reverse sections." Let us take the order 1,2,3,...,9 as the original order. In this case all the signs would be <. For a given arrangement of the relation signs underline the relation signs which are in the reverse order, so they are >. Also we underline a circle if next to it there is an underlined sign. This way we get sections, some of them are underlined and some of them are not.



Take the original order, and in each underlined section put the numbers in reverse order.



(iii) "Contour-map" (or "relief"). Imagine the problem as mountains and valleys. Take the first circle. If the next sign is <, then draw the next circle higher, if it is > then draw it lower. With dotted lines we indicated the levels. Now put the numbers from 1 to 9 from the bottom level to the top. (Within a level the order is not important.)



(iv) "Just different." Take the first circle and write in it any number. Then take the next one and write a new number in it which satisfies the relation before this circle. The following figure shows such a "solution":

Then put these numbers in increasing order, and take the ordinal number of them.

-9	-4	-2	3	7	12	42	85	93
1	2	3	4	5	6	7	8	9

Replacing the numbers by their ordinal number clearly we get a good solution. In our case we get:

$$12 < 42 < 93 > -4 < 7 < 85 > 3 > -2 > -9$$

$$6 < 7 < 9 > 2 < 5 < 8 > 4 > 3 > 1$$

Remark: Now we can see how important is the step-by-step building. Problem 2 was easy but it helped a lot to get problem 3.2 which is a real task. Without problem 2 it is even harder to ask problem 3.2; we have to explain to many of the students what do we mean by the question. Following our track the question was natural, and the different solutions -which I have learned from my students- are beautiful.

The induction type solution is common in mathematical textbooks, but at the age of 12-14 it is not the way children would do the proof. This method gives just one solution.
The contour-map solution gives us more freedom. At a certain step we may move any number of levels up or down, within a level we may permute the numbers. Looking back to problem 2 the contour-map idea gives a new perspective. Let me mention that the last solution I have heard from a boy who longs for informatics. It combines the algorithm with different random numbers and ordering.

To have different solutions of a problem is one of the beauty of mathematics. It might be an important indicator of ability to do mathematics and also a way to improve knowledge and problem solving skills (Ma, 1999. p. 140).

Combinatorics

Problem 4. How many ways can we put the integer numbers from 1 to 6 in the circles according to the given relation signs?

$$\bigcirc < \bigcirc < \bigcirc < \bigcirc > \bigcirc > \bigcirc$$

Solution: The place of number 6 is determined, it must be in the fourth place. Out of the remaining five numbers choose three and write them in increasing order before 6, the last two numbers write in decreasing order after 6. So there are $\binom{5}{3} = 10$ solutions.

Remark: Here we started something which could be continued by other arrangements of the relation signs. The problem is easier if the circles are along a line like in problem 3 but it can be quite hard if the figure is more complicated like in problem 2. With such a complex figure the solution may lead to a long process: meticulously set out the many cases.

Problem 5. Find the number of positive integers whose digits are strictly decreasing from left to right.

Solution: We have 10 digits altogether. Taking any subset of these determine a unique number with the required property. Consider all the 2^{10} subsets of these 10 integers, and check, which of them give good solutions. The empty set is a subset but does not determine a number, and the one-element subset consisting of 0 alone does not determine a positive integer. So the answer is $2^{10} - 2 = 1022$.

Remark: This approach of the problem gives a quick answer. During my teaching experience I met several times another method: count first the numbers with one digit, then with 2 digits, 3 digits and so on. This way we get the following result:

$$9 + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} + \ldots + \binom{10}{10}$$

If we have worked previously with our students on Pascal's triangle or they are familiar with the binomial theorem then this sum is not frightening but a good old friend.

Algebra

When a child gets acquainted with numbers, first come the positive integers. Out of two different positive integers it is very rare that someone does not know which one is greater. A little bit harder is if we have negative numbers, some of my students hesitate a little when I ask which one is bigger -34 or -5?

The situation is more complicated with fractions. To compare them is not so easy. In problem 6 I am going to look at various stages of this, starting with easy questions and then turning to harder ones involving larger numbers, negative numbers and powers.

<u>Problem 6.</u> Put the relation sign between the two numbers:

(6.1) $\frac{1}{6}$ or $\frac{1}{7}$ - basic understanding of the concept of fraction

(6.2) $\frac{2}{9}$ or $\frac{3}{8}$ - use the form with a common denominator

(6.3) $\frac{1234}{3456}$ or $\frac{2341}{4563}$ - estimation, comparing both with $\frac{1}{2}$ gives a quick answer

(6.4) $\frac{321}{654}$ or $\frac{312}{654}$ - same denominator, compare the nominator

(6.5) $\frac{321}{654}$ or $\frac{321}{645}$ - same nominator, smaller denominator means larger fraction

(6.6) $\frac{321}{654}$ or $\frac{322}{655}$ - this example is an invitation to compare fractions in the form $\frac{a}{b}$ or $\frac{a+1}{b+1}$; or/and look at such sequences as $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},...$ with difference $\frac{1}{6},\frac{1}{12},\frac{1}{20},...$

(6.7) Put the four fractions in increasing order: $\frac{22^2}{33^3}$; $\frac{-22^2}{34^3}$; $\frac{(-22)^2}{35^3}$; $\frac{22^2}{(-36)^3}$.

Problem 7. (7.1) In one of his pockets Bob has coins of value 10 Forints, in another one only coins valueing 100 Forints. From one pocket he takes 3, from the other 7 coins. How shall he do it in order to take more money out of his pockets?

(7.2) We know that $a<b<c$ and $x<y<z$. Is it possible to determine, which one is smaller $ax+by+cz$ or $ay+bz+cx$?

(7.3) Two real numbers a and b are given such that $0<a<1$, and $b<-1$. Put 1, b, b^2 and b^3 into the dotted places so that the value of the following expression should be maximal:

$$1\cdot ... + a\cdot ... + a^2\cdot ... + a^3\cdot ...$$

Solution: Usually my students laugh at me, they find this question so easy: 3·10 + 7·100 is more than 7·10 + 3·100. The interesting thing is that not much more is needed for the rearrangement theorem. The two other questions of problem 7 help to get closer to this theorem. So using two times the idea of (7.1) $ax + by + cz > ay + bx + cz > ay + bz + cx$. Because in (7.3) $1 > a > a^2 > a^3$ and $b^2 > 1 > b > b^3$, the expression will be maximal in the case $1\cdot b^2 + a\cdot 1 + a^2\cdot b + a^3\cdot b^3$.

Remark: If our group is more confident with numbers and the use of letters as variables may cause problem, then we might substitute the variables with given numbers and ask the question like that.

Graphs and geometry

In problem 2 and problem 3 we have seen figures which might be generalized as follows. The circles are vertices of a graph, between some vertices there is a directed edge, pointing from smaller to larger. This leads to the following question:

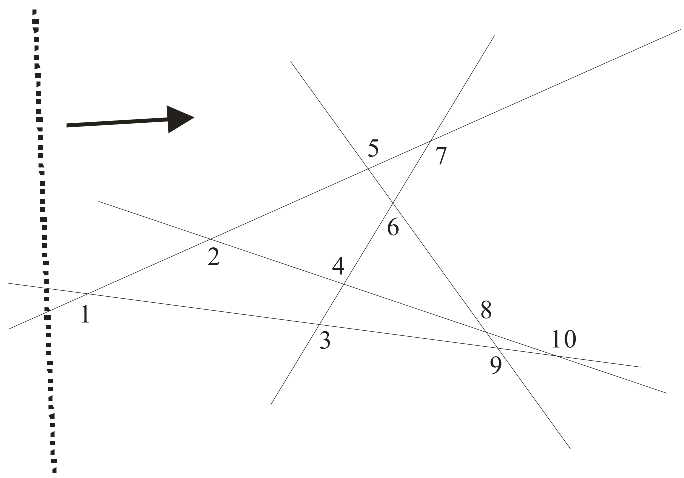
Problem 8. Let G be a directed graph with n vertices. Number the vertices with the positive integers from 1 to n so that the arrow of each directed edge should point from a smaller to a larger number.

Solution: Although the question seems to be more serious now, we have already solved it. Problem 2 taught us that there is no solution if there is a directed circle within the graph. Problem 3 taught us how to solve it otherwise. For example use induction on *n*, for the induction step we take either a sink (a vertex with only entering edges) and write there the largest number or a source (a vertex with no entering edges) and the smallest number. The remaining part of the graph is done by the induction hypothesis.

Remark: Several times I have done this problem in class using the graph of clothes. Each item is a vertex (socks, t-shirt, trousers, etc), the directed edge shows which one should be put on earlier.

<u>*Problem 9.*</u> There are *n* lines in the plane. Put different integers to the intersections so that the numbers should be in monotone order along any line.

Solution: Take an extra additional line which is not parallel to any of the lines which go through at least two of the intersection points. Sweep the plane with this line, the intersection points will get on the line one by one, in this order they get the numbers 1, 2, ...



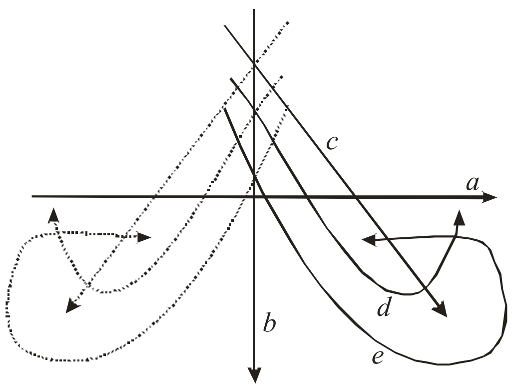
Remark: We might use the idea, we have learned in problem 3. Put our lines in the coordinate system so that none of the lines which go through at least two of the intersection points is perpendicular to the *x* axis. Then the *x* coordinates of the intersection points are different real numbers, so the same idea as in problem 3.(iv) gives a solution.

Let us transform problem 9. Instead of lines we might define pseudolines. They are simple curves, without self intersection, any two might have at most one intersection, at any intersection they must cross each other as lines and cannot just touch each other like circles. The first mathematician who investigated them and used the word pseudoline was F. W. Levi (Levi).

Definition: Take a pseudoline and two points of it. The closed part of the pseudoline which is between the two points we call a pseudosegment.

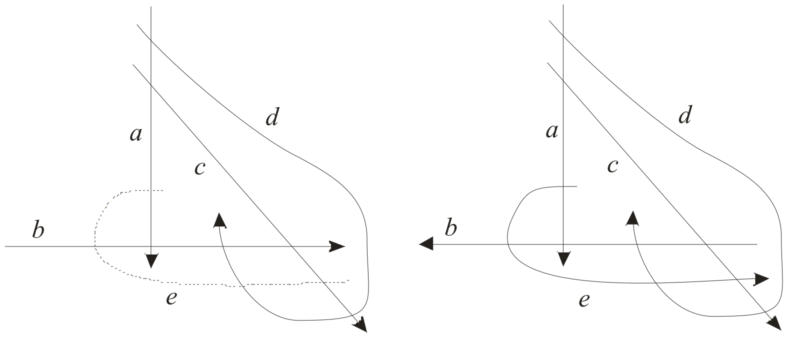
<u>*Problem 10.*</u> There are *n* pseudosegments in the plane. Put different integers to the intersections so that the numbers should be in monotone order along any pseudosegment. Is this always possible?

Solution: We will see that such a numbering is not possible for the figure below. Suppose that one could have written the numbers, then we draw an arrow along each pseudosegment which goes from smaller numbers to bigger ones. We will show that for any arrangement of the arrows we get a directed circle which makes the numbering inpossible.
Without loss of generality we may assume that along *a* the arrow is from left to right. If *b* is directed from top to the bottom then we use *c*, *d* and *e*, otherwise we use the remaining three pseudosegments. These two cases are symmetric so we investigate the first one.



If any one of *c*, *d* and *e* would have other direction than the one indicated on the figure, then that pseudosegment with *a* and *b* would generate a directed circle. In other words: if we want to avoid a directed circle, then the direction of *a* and *b* determines the direction of *c*, *d* and *e*. But also these three (*c*,*d*,*e*) pseudosegments form a triangle and we get a directed circle along this triangle.

This special figure shows that the behaviour of pseudosegments is different from the normal segments. We have used 8 pseudosegments to get such a figure which forces us to get a directed circle. The natural question can be asked whether could we draw such a figure with less pseudosegments? I have created the following figure with 5 pseudosegments:



case 1. case 2.

Without loss of generality we may assume that the direction of *a* is from top to bottom. Then look at the direction of *b*.

case 1: If it is from left to right then take the triangle formed by *a*, *b* and *c.* Either *c* is directed according to the figure, or we get a directed circle. The same holds for *d*. But now the triangle of *b*,*c* and *d* forms a directed circle.

case 2: If *b* is directed from right to left, then either *e* is directed according to the figure, or *a*,*b* and *e* determines a directed circle. Now we might repeat the argument we had at case 1 for the pseudosegments *a*, *c*, *d*, *e* since *e* plays the role of *b*.

Remark: It is always possible to direct four pseudosegments in such a way that no directed circle occurs. To prove it is less interesting then the surprising result that there is such a special figure with 5 pseudosegments.

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Problem Orientation within Teacher Education
A Report about Experiences

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Abstract

To promote problem orientation and self-activity of pupils first the teachers have to make experiences with problems by themselves. Here I will report on three seminars for pre-service teacher students focussing on problem orientation. First there will be given themes students did work on. Then ways teacher students worked on three given problems will be shown whereat the emphasis lies on different approaches the students had chosen.

Introduction

Problem orientation is an important aspect within mathematics education we all know. But we also know that in school reality problem orientated mathematics teaching with self-activities of the pupils as well as finding and discussing different ways of working on a problem you do not find often. I think besides theoretical discussions within the didactical community and the presentation of interesting proposals for different classroom conditions - which are important - first of all the teachers must be familiar with and get a positive belief about problem orientation in mathematics education.

Survey of three seminars

Besides integrating some theoretical and practical aspects about problem orientation into all my lectures and seminars within the teacher education at the University of Bielefeld in the last two years I offered three special seminars with about twenty students per seminar concentrating on problem orientation in mathematics education. Two of these seminars have been seminars for preparing teacher students for writing a final Bachelor paper referring to problem orientation. The other seminar was a normal one for senior teacher students. In all of these three seminars on one hand I made some inputs with copies out of literature and on the other hand the students had to work by themselves with different problems and find new problems within the discussed problem field. - I here already will mention that the last task was very hard for the students.

Before going into details I would like to give an overlook of the topics we discussed.

In all of the three seminars in the first session (lasting one and a half hour) I presented three different problems to work on by themselves (together with their neighbours) without giving any help or hint. I will report on the results of this session later on.
In the second session (be geared to Pehkonen & Graumann 2007 and Büchter & Leuders 2005) I first made an oral input about the

- history of problem orientation as well as learning by discovery, learning by doing and self regulation.

After that we discussed some papers concerning

- definitions about problem, problem solving, heuristics and problem orientation.

Also by means of copies out of special literature we discussed

- types of tasks, types of problems and ways of developing tasks by ones own.

In the normal seminar we deepened theoretical aspects like

- variation of tasks (according to Schupp 2004)
- "logic of failure" (according to Dörner 1989),
- mathematical learning from constructive view,
- beliefs about problem orientation,
- barriers in respect to changing mathematics teaching,
- aims of and motivation for problem orientation and
- statements according to problem orientation in official guidelines.

Mathematical topics considering the aspect of problem orientation have been:

- figured numbers and special sums,
- sequences and chains of numbers,
- Pythagorean triples,
- triangles with integers as side length,
- regular polygons and polygons in space as well as
- problems from PISA.

In the seminar focussing on a Bachelor paper besides formal aspects and discussions about analysing and planning teaching each student had to bring to discussion his/her theme and schedule of his/her paper. The students could choose their approximate theme out of a list I did give to them. About one third of the students took a theoretical theme the others took a theme for making experiences with children.
The topics used as theme have been:

- *Problem solving and heuristic with Polya*
- *Learning by discovery in general education and mathematics education*
- *Problem orientation and constructivism in mathematics education*
- *Variation of tasks as methodical way in mathematics education*
- *Activ-discovery Learning and productive exercising in mathematics education*
- *Self-activity and self-regulation in the discussion of didactics of mathematics within the last ten years*
- *Partition of sets and representations of numbers as sums in grade 1*
- *Number trains concerning products of digits in grade 2*
- *Magic squares and sudoku - a topic for grade 2 and 4*
- *Number walls in grade 2 and 4*
- *Number chains in grade 3*
- *Distribution of prime numbers in grade 3*
- *Polyominos – a geometrical problem field for grade 3*
- *Fermi tasks in grade 3 and 4*
- *Division with rest in grade 4*
- *Huge numbers and data - a topic for grade 4*
- *Arithmetic stories in primary school*

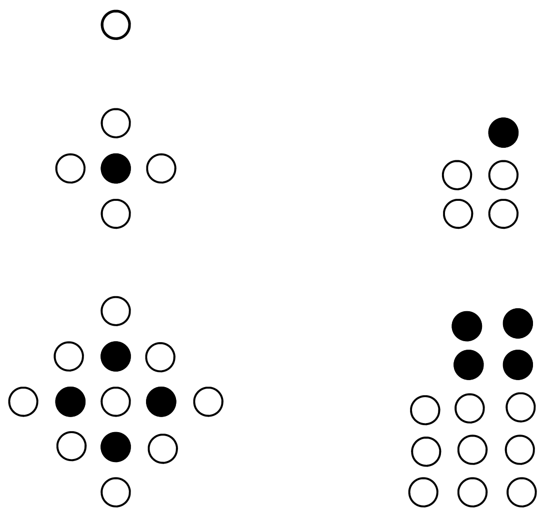
The "Mason problem" as inspirer

Some years ago I did hear from a seminar in Debrecen where John Mason as guest was present. In this seminar John Mason asked the participants (mathematics teacher students and secondary mathematics teacher) to solve the following problem.

$1 =$

$1 + 3 + 1 =$

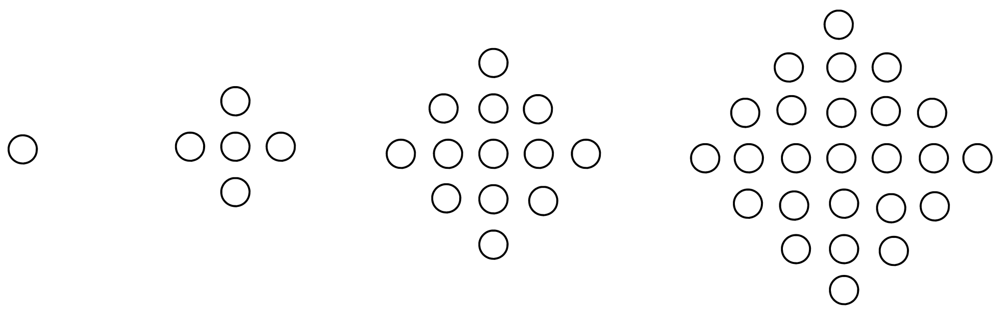
$1 + 3 + 5 + 3 + 1 =$



Task: "*Formulate a general question for this problem! Try to formulate a conjecture to your question! Prove the conjecture!*"

After 10 to 15 minutes it was clear that such open problems are very uncommon to Hungarian students and teacher, most of them could not do anything. This caused me to present the "*Mason problem*" to my students at the beginning of the seminars in terms of not influencing them by discussions within the seminar (even though the title of the seminar may have influence and in their first semester they have already seen simple figured numbers like square numbers and triangle numbers). I wanted to know how they will act on this problem.
I varied the presentation of the Mason problem a little bit in form of not giving symbolic hints in respect to sums of numbers and not painting the little circles different because I wanted to see how the students will do it by themselves and whether they will discover different structures.

Two rows of the figures shown below have been given with the following text: "*Draw the next two figures of this sequence. Look out for partial figures and mark them. Which arithmetical representation do you can find on this way?*"



I also did add two other problems of a different type because the students should make experiences with different types of problems too. These two other problems have been given in text in the following way.

Problem concerning a ghost of a river: *A ghost of a river says to a walker who just will cross a bridge: "If you cross the bridge I will double the money you have in your pocket; but if you go back across the bridge I will take 8 Euros away from your pocket." When the walker came back the third time the money in his pocket was gone away (exactly 0 Euros).*

Problem concerning small animals: *Grandfather Miller has in his yard hens and rabbits. Once upon a time he counted 7 heads and 20 legs. (Variation: He did count only 20 legs)*.

Different results of these two problems given in text

First of all I can tell that nearly all students reported (in the discussion at the end of the first double hour) that an open problem like that from Mason was very new for them. But all of them started to work on that problem and most of them got a special result (at least together with a neighbour). It also could be noticed that in any of the seminars there appeared different ways of working at the problems.

In the following I will present all different ways of working with these problems; in doing so I will start with the two problems given in text.

Problem concerning a ghost of a river

1. Method of trial and error with variation: We start with 6. The transformation from this are $6 \to 12 \to 4 \to 8 \to 0 \to 0 \to$ not possible. We see that we have to get 6 after the first crossing and way back. Thus we try it with two more Euros and get $8 \to 16 \to 8 \to 16 \to 8 \to 16$ and find an endless sequence. Now we try the number between and get the solution $7 \to 14 \to 6 \to 12 \to 4 \to 8 \to 0$.

   [By varying our thoughts we could start with 5 or 4 and see that all numbers of the sequence decrease. We also can see that starting with 9, 10, … will let increase all numbers of the sequence. That means we may find a functional relation.]
2. Method of working backwards: $0 \leftarrow 8 \leftarrow 4 \leftarrow 12 \leftarrow 6 \leftarrow 14 \leftarrow 7$.
3. Method with algebraic formula: $2 \cdot (2 \cdot (2 \cdot x - 8) - 8) - 8 = 0$ or

   $x \to 2x \to 2x - 8 \to 2\cdot(2x-8) \to 2\cdot(2x-8)-8 \to 2\cdot(2\cdot(2x-8)-8) \to 2\cdot(2\cdot(2x-8)-8)-8$

   From $2\cdot(2\cdot(2x-8)-8)-8 = 0$ we will get $x = 7$.

   [We can get a generalisation via this method with $2 \to a$, $8 \to b$ and $3 \to n$:

   $x \to ax \to ax-b \to a\cdot(ax-b) \to a\cdot(ax-b)-b \to a\cdot(a\cdot(ax-b)-b) \to a\cdot(a\cdot(ax-b)-b)-b$

   $\to a\cdot(a\cdot(a\cdot(ax-b)-b)-b) \to a\cdot(a\cdot(a\cdot(ax-b)-b)-b)-b \ldots \to a^n x - (a^{n-1} + a^{n-2} + \ldots + 1)\cdot b$.]

Problem concerning small animals

1. Method of trial and error with variation: We try 4 rabbits → 16 feet, with the left 4 feet we get 2 hens; that make together 6 heads. Because one head is undercharged we have to increase the number of hens respectively decrease the number of rabbits. 3 rabbits → 12 feet with left 8 feet and 4 hens → 8 feet gives 7 heads and 20 feet as desired.
2. Method of working backward from the heads: Any animal has at least 2 feet, so with 7 heads we have at least 14 feet. The rest of 6 feet is going in pairs to 3 rabbits, so we get 3 rabbits and 4 hens.

3. Method with algebraic formula: x = number of rabbits, y = number of hens. $4x + 2y = 20$ (number of feet) and $x + y = 7$ (number of heads) . Then solving with algebraic instruments gives $x = 3$, $y = 4$.

Variation of this Problem

The variation shall show to the students that we also can get a problem that has more than one solution and we have to find a systematic for finding all solutions.

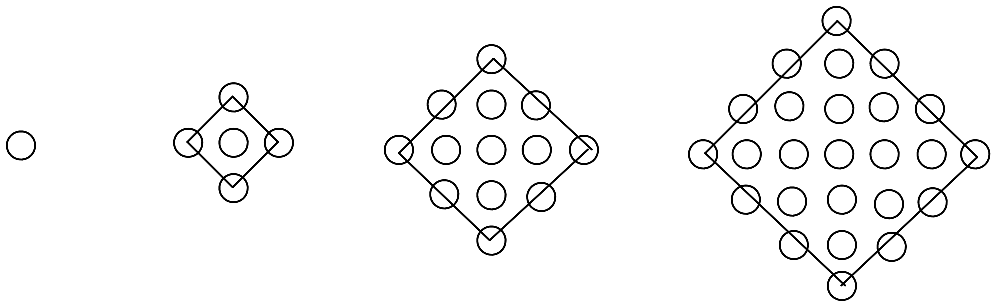
Here the minimal number of heads is 5 because 5 rabbits and 0 hens makes 20 feet. Reducing the number of rabbits step by step you will get 6 heads (4 rabbits and 2 hens), 7 heads (3 rabbits and 4 hens), 8 heads (2 rabbits and 6 hens), 9 heads (1 rabbit and 8 hens), 10 heads (0 rabbits and 10 hens). The solution with 0 hen and that one with 0 rabbit probably does not fit to the text and thus these solutions have to be erased.

In addition we can make investigations in respect to functional relations like "Reducing the number of rabbits by one causes increasing the number of hens with two" or "Reducing the number of heads by one causes decreasing the number of hens with two".

Different ways the students worked with the "Mason Problem"

The following different groupings by colouring some circles or combining some circles with a line and symbolic descriptions have been the following:

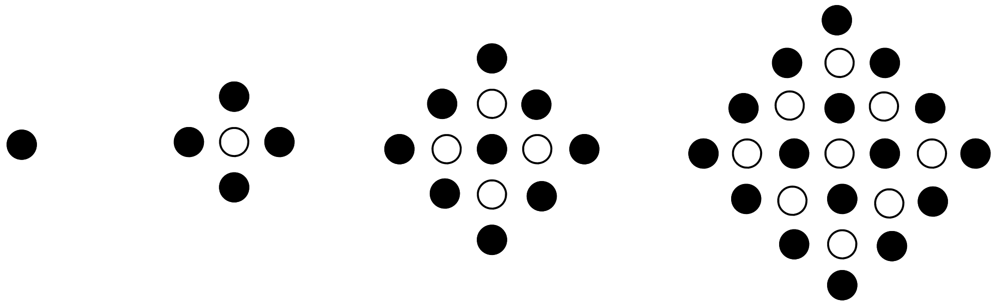
1. Combine with a line the circles building the frame of the figure. So you



   get the description 1, 1+4, 1+4+8, 1+4+8+12, … and in general
   $1 + 4 \cdot (1 + 2 + 3 + \ldots + (n-1))$.
   If we already know that $1 + 2 + 3 + \ldots + (n-1) = \frac{1}{2} \cdot (n-1) \cdot n$ we will get the general symbolic description $1 + 2 \cdot (n^2 - n)$ [resp. $2n^2 - 2n + 1$].

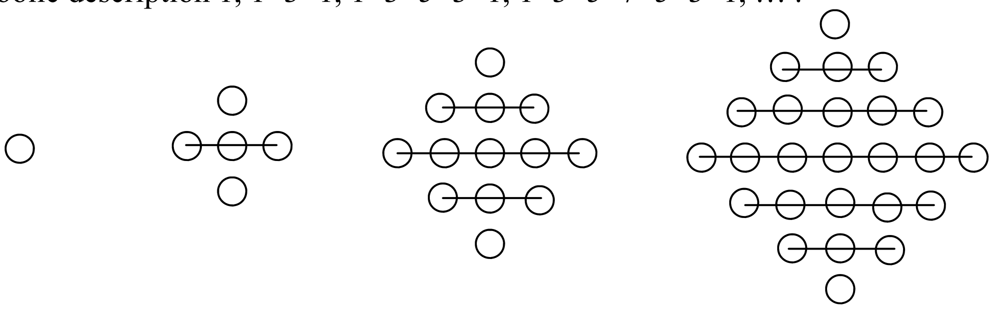
2. In a more arithmetical view on these figures some students looked at the total number of circles in each figure: 1, 5, 13, 25, … . From this they detected that the difference sequence 4, 8, 12, 16, 20, … is built by the multiple of 4. On this way they got the same general description 1, 1+4, 1+4+8, 1+4+8+12, … resp. $1 + 4 \cdot (1+2+3+\ldots+(n-1))$.

3. Some students coloured the circles in the frame together with inner circles for getting a squared number. The non-coloured circles then built a squared number too but a smaller one, more precisely the length of the side is one less than the length of the side of the coloured square.



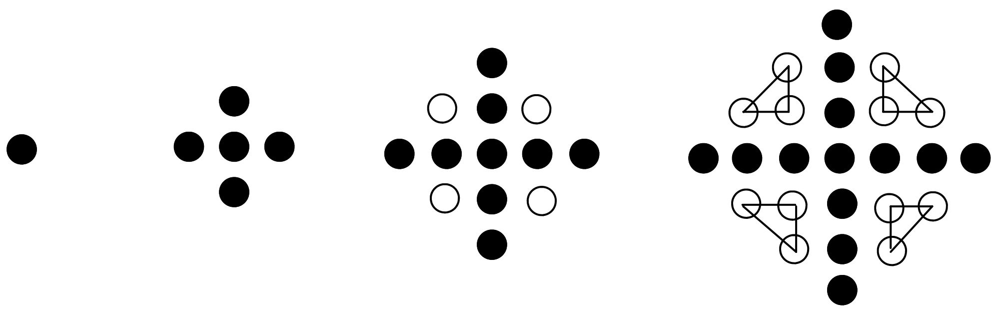
The symbolic description thus came out as 1^2, 2^2+1^2, 3^2+2^2, 4^2+3^2, … or in general $(n-1)^2 + n^2$ [respectively $2n^2 - 2n + 1$].

4. A fourth group of students looked at the horizontal (or vertical) rows and got the symbolic description 1, 1+3+1, 1+3+5+3+1, 1+3+5+7+5+3+1, … .

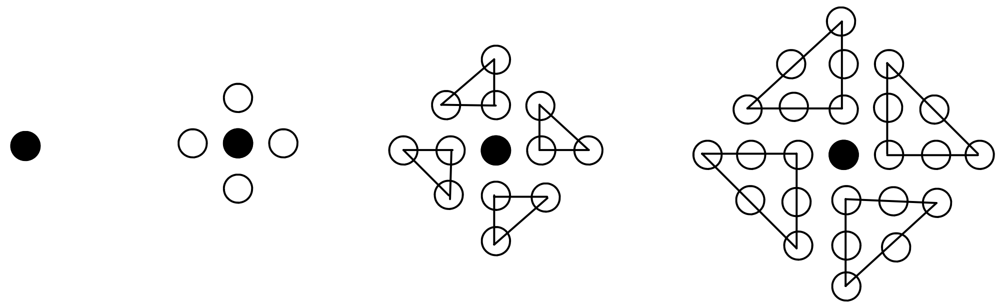


If we already know that the numbers 1, 1+3, 1+3+5, 1+3+5+7, … describe a square number we can see the identicalness with the symbolic descriptions above.

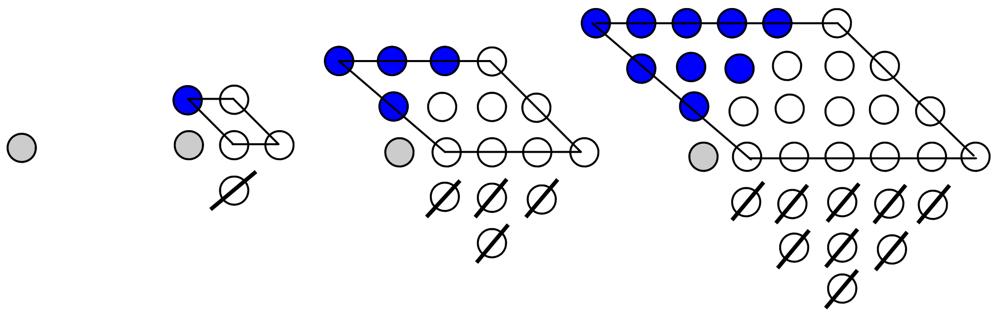
5. One student coloured the vertical and horizontal middle lines building a cross. The non-coloured circles then build four triangle figures and we get 1, $1 + 4\cdot1$, $1+ 4\cdot2 + 4\cdot1$, $1 + 4\cdot3 + 4\cdot2 + 4\cdot1$, … resp. $1 + 4\cdot(1+2+3+\ldots)$.



6. Two students also saw four triangle numbers but including always one branch of the cross so that only the middle point is extra standing. This leads to the formula 1+4·[1+2+3+…+(n-1)] directly.



7. Another group of students resorted the figures of circles by erasing the rows below the horizontal middle line and then added these circles on the left side of the remaining circles so that after that all horizontal line have the same length without the bottom line which has one circle more. So they got the sequence 1, 1+ 2·2, 1+ 3·4, 1+ 4·6, 1+ 5·8, 1+ 6·10, … .



A general description they did not find because it is no so easy as before. With some considerations you can find the formula 1+ n·(2·(n-1)) resp. 1+ 2n² - 2n.

In the discussion with the whole group the different solutions were presented with adding the missing symbolic descriptions. For teacher students this is very important because they can see the large variety of working on such a problem. Later on as teacher in school on one hand it is important to be open for different ideas of pupils and on the other hand the arrangement for working with problems should include working in small groups as well as reflecting different approaches and their connections.

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Primary school children's model building processes by the example of Fermi questions

Jana Henze & Torsten Fritzlar

Abstract

The development of modelling skills is considered an important goal of mathematics education, also at primary school. In this context, manifold potentials are ascribed to so-called Fermi questions. We therefore emphasise the importance of further exploring how primary school pupils actually handle these kinds of problems.
In this article, with reference to a case study, we develop a suggestion for a descriptive progression model of work processes that take place when dealing with Fermi questions. This model combines aspects of both modelling and problem solving in a fruitful way.

1. Introduction

Model building processes play a major role in current discussions and research on mathematics education (e. g. Blum, Galbraith, Henn, & Niss, 2007; Kaiser, Blomhøj, Sriraman, 2006; Sriraman, Kaiser, Blomhøj, 2006), and adequate competencies of students are an important goal of mathematics classes, as stipulated for example in German education standards or in the context of the PISA-study's *Mathematical Literacy*-concept (OECD, 2004). However, the reader may ask herself why this topic is picked up within the context of this volume on problem solving.

We believe that there are several similarities between problem solving and modelling (also e. g. Niss, Blum, & Galbraith, 2007), and that both approaches can be combined in a fruitful way (e. g. Greefrath, 2008). This becomes even more obvious when taking a closer look at the processes involved, or when comparing different process modellings as depicted in existing literature.
Describing model building processes is usually done by means of a cycle; Figure 1 shows a typical version by Blum (1985). This cycle begins with the actual situation which must then be structured, simplified and idealised by identifying relevant pieces of information with regard to the problem statement. Of course, the real model thus created retains a subjective touch, amongst others due to the individually available mathematical tools that, during the following step, can be used to develop a mathematical model that must fit to the real model. Here, different types of mathematisations are often possible. The formulation of a mathematical solution is often the easiest step, also because the mathematical or real model was fre-

quently developed under consideration of one's own personal skills. In a final step, the mathematical results must be revised, or applied to the actual situation. Should they prove useless, models have to be amended and the cycle repeated.[1]

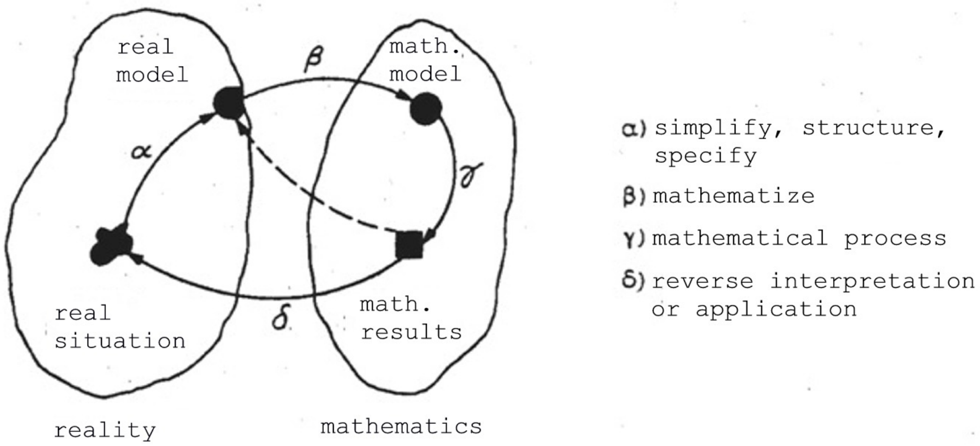

Figure 1: Cyclic model of modelling by Blum (1985)

If you compare the cyclic model of modelling with a progression model for problem solving – e. g. by Pólya (1971) – you will encounter some parallels that are schematically visualised in Figure 2 and according to which modelling processes can be understood as one specific type of problem solving.

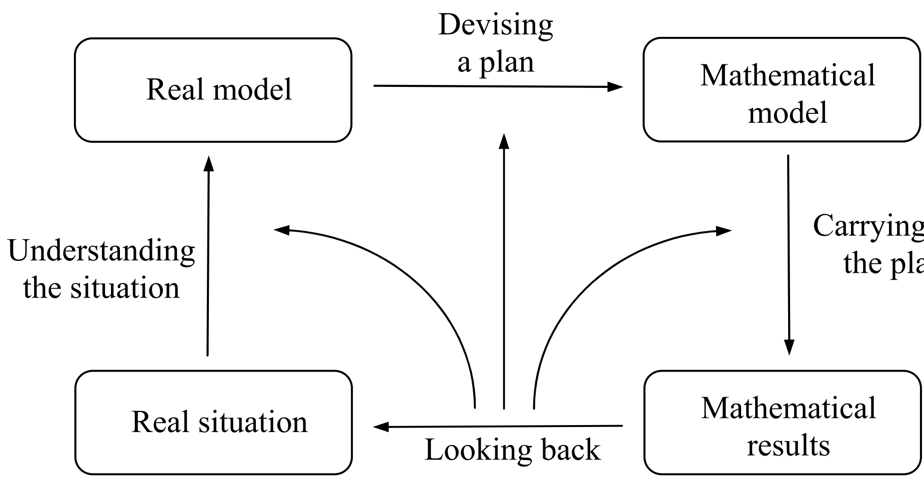

Figure 2: Modelling and problem solving

2. Theoretical background

According to German education standards, modelling is a general mathematical competence to be developed also at primary schools (Kultusministerkonferenz, 2005).

[1] In newer models of the model building process, some elements are elaborated further. A differentiation is frequently made between the actual situation and its mental representation as the starting point of modelling (in a narrower sense) (e. g. Borromeo Ferri, 2006).

The following example can be found in an explanatory publication (Walther, van den Heuvel-Panhuizen, Granzer, & Köller, 2008) to illustrate this area of competence (p. 35):

4000 pupils in 48 classes
Gevelsberg - Summer holidays are coming to an end. The large number of children walking to school are a clear indication that Gevelsberg's 9 schools have once again opened their doors.
This year, almost 4000 pupils are enrolled in a total of 48 classes. For some children, the summer break was much too short, but most of them are looking forward to the new school year.

No, there are no classes of 80 pupils.

Figure 3: Proposed modelling task

According to the authors, to solve this exercise, pupils must extract relevant information and neglect other data; translate a realistic problem into mathematical terms and develop a mathematical model; solve it inner-mathematically; and finally refer back to the initial situation (cf. Figure 1).
However, I think this task is a fairly simple one, as the pupil working on it only has to choose two out of three given figures (both of which are included in the heading of the newspaper article), and only make one calculation to come to a basic "yes" or "no" conclusion.

Contrary to this task, „good“ modelling problems that require actual model building (and problem solving) can be described by attributes such as *realistic, data-based, complex, open, differentiating*, while these characteristics are not independent of each other and the fifth rather refers to the possible use of the problem in class (Henze, 2009). Model building processes thus consist less of neglecting empirical details until the "skeleton", which they created, can be translated into mathematical terms, but rather of a *structural extension* of the situation, of introducing new elements and (mathematical) relations (Schwarzkopf, 2006).

Is it possible to successfully employ such “good” modelling problems, enabling authentic mathematical activity, already at primary school? One possibility to do this is seen in so-called Fermi questions[2] (Kaufmann, 2006; Peter-Koop, 2004, 2008; Wälti, 2005) which are problems providing no or insufficient information for a calculational solution and for which the goal is to define a suitable solution range through justifiable assumptions (Kaufmann, 2006).

The high potential that can be seen in Fermi questions when used in class makes it plausible to us to further examine the question in which way students actually deal with these kinds of challenges. However, the cycle in Figure 1 is hardly a suitable point of departure for developing a descriptive model of relevant work processes. For at primary school it is rarely the case that, when solving a modelling task, the factual context must temporarily be left completely (in the sense of the modelling cycle); that, so to speak, a „pure“ inner-mathematical problem is created. One rather makes the effort to ensure that the children do not lose connection with the issues at hand even though they are busy with their calculations (Schwarzkopf, 2006).

The descriptive system by Möwes-Butschko, on the other hand, could prove more useful. Its focus is rather on individual processes and was employed by the author to describe the handling of “open realistic tasks” by primary school pupils. Figure 4 shows one example of the used exercises combining text and pictures (Möwes-Butschko, 2007).

How tall is the baby elephant?

Figure 4: An “open realistic task” used by Möwes-Butschko

Möwes-Butschko suggests the following categories to describe the pupils’ work processes (without further explanation): *Orientation, Planning, Data collection, Data processing, Data securing, Argumentation, and Control.*

3. Research question

Within the frame of a teaching experiment, Fermi questions (without pictures) were employed in two different Year 4 classes. The idea was to explore whether the pupils’ work processes

[2] Named after Enrico Fermi (1901 – 1954), a prominent nuclear physicist of the 20th century who repeatedly gave such exercises to his students. A typical example, ascribed to Fermi himself, is the question of the number of piano tuners in Chicago.

can be described by the categories suggested by Möwes-Butschko, and in how far a further differentiation of these categories seems appropriate. Another aim was to gain further indications regarding the frequently postulated potential of Fermi questions in modern mathematics education.

4. Implementation of the study

For the case study, one Year 4 class was chosen from each of two primary schools in the German federal state of Lower Saxony. The lessons were held shortly before the end of the primary schooling period, so that we could expect the pupils to possess the knowledge and competencies set by the curriculum.

Pupils in Class A were used to a very traditional teaching style regarding content and methodology and did not have any experience in working on modelling exercises. Class B was also not familiar with modelling exercises; they had, however, worked in groups before.

The experiment comprised a total of four lessons. During the first lesson, pupils were able to gather first experiences with Fermi questions, and possible approaches were discussed in class. To us, it was important that pupils understood and accepted that there is not one definite and exact solution to the problem, that you must frequently estimate, that there are several possible ways to solve the problem, and that various auxiliary tools may be used (research, expert interviews,...). A fixed problem solving scheme such as in Figure 1 was not discussed, however.

During the second and third lessons, the pupils dealt with the following Fermi questions: "How many worksheets do you complete during your time at school?" and "How much time of your life do you spend brushing your teeth?" Results and work processes were to be recorded on a group poster for the following presentation and discussion of results.

Two groups of pupils were chosen from each class and filmed while working on their first Fermi question. When choosing the group members, attention was paid to the fact that these pupils worked together well and could easily pick up a conversation. The performance level of individual pupils was not relevant.

Finally, during the fourth lesson, the pupils developed their own Fermi questions.

When selecting the Fermi questions, we wanted to make sure that most pupils would be able to handle the (obvious[3]) mathematical demands, and that the context of the questions would be clear to them. By addressing pupils individually ("How many worksheets do *you* complete

[3] From the point of view of the persons designing the question.

during your time at school?”), they were encouraged to contribute by sharing their own experiences.
Data collection was crucial for the first question, while processing this data should then be a fairly simple exercise. To solve the second question, however, data collection played a minor role, while dealing with different time units was the trickier task to tackle for primary school pupils. Both questions demanded strong argumentation skills.[4]

In a first step, the videotaping was transcribed. For one group of pupils, the transcription was interpreted through qualitative content analysis by two reviewers independently of each other. The aim was to trace the work process of the *group*. The setup of this analysis was generally open and gave room for creating categories, whereas Möwes-Butschko’s findings and basic considerations on modelling and problem solving served as the general points of departure. The devised categorisations were then compared, differences discussed and subsequently, joint decisions regarding their allocation were made. The following step was an open analysis of the other groups’ work processes. Repeated rounds of analysis served to revise all categorisations and introduce refining sub-categories where suitable.

5. Results

Due to the limited scope of this paper, we can only discuss the modelling processes of two groups of pupils in more detail. Yet, in order to give an idea of the broad spectrum that was to be observed, we will present two very different work processes, both from class B.

Two of the three boys in Team B1 will be attending „Gymnasium“ (grammar school)[5] after the summer holidays, so their grades in mathematics are accordingly high. The third boy has received a recommendation to continue secondary schooling at „Realschule“, yet, he is very good at mathematics. All three of them are rather reserved compared to the two girls and two boys in Team B2. One pupil in this group is repeating Year 4, and his grades in mathematics are average, like one other pupil’s in his team. The other two generally perform very well.

The first team is looking for an answer to the question of how many worksheets have to be completed during time at school.
After the teacher had presented the question to the plenum, the pupils tried to gain their own

[4] The following questions must be answered, amongst others: How many school years and subjects are taken into consideration? How does the number of work sheets change, respectively? Starting from which age to which final year, how many times daily and for how long do you brush your teeth?
[5] In Germany, pupils are allocated to one of four types of secondary school depending on their performance and abilities shown in primary school: Gymnasium (8 years, leads up to university education), Realschule (6 years, intermediate level), Hauptschule (5-6 years, prepares for vocational training); Gesamtschule combines the three types in one institution.

understanding of the problem within their group and spontaneously gave their opinions (*Orientation – understanding the problem*[6]). Following an initial phase of cluelessness, the boys began to develop first ideas – it was suggested, amongst others, to go through all folders of previous school years and count the worksheets filed there –, which were not picked up again or elaborated further, though (*Orientation – first ideas*). They later came up with the idea to estimate the respective number of worksheets per subject per school year – an approach that seemed practicable to the pupils (*Planning*). They started making a list of the subjects taught at primary school (*Collecting data)*. When estimating the number of worksheets per subject (*Collecting data, Estimations*), it partly came to longer discussions (*Collecting data, Discussion*), after which the total number for all 4 years of primary school still had to be calculated (*Processing data, Calculations*). A longer working phase thus resulted in a first answer to the initial problem statement.

In a following short conversation with the teacher, the pupils first of all reported on their approach and their current work progress (*Report to external party/Ask for help*). They also discussed whether secondary schools should be taken into consideration when answering the Fermi question. After the pupils had agreed on this, they commenced a further working phase very similar to the one just described. However, there were longer discussions regarding the type of secondary school to be examined (6 years of „Realschule“ or 8 years of „Gymnasium“, see footnote 5), and there was some insecurity regarding the list of subjects (number of foreign languages).

The pupils worked on the problem for a total of 32 minutes in a motivated manner (which very much surprised the teacher, who was at first skeptical). In the process, each of them was able to contribute his ideas, there were intensive discussions regarding the approach to be taken, the list of subjects and the respective number of worksheets. Phases of data collection and -processing were very extensive, calculative skills were intensively practiced during the latter ones.

Figure 5 shows the poster on which the pupils recorded their results, which they proudly presented at the end of the lesson. The high total number of 6160 worksheets may be surprising, but we believe it is also an indication of the impression pupils get of their lessons.

[6] See following section on category system

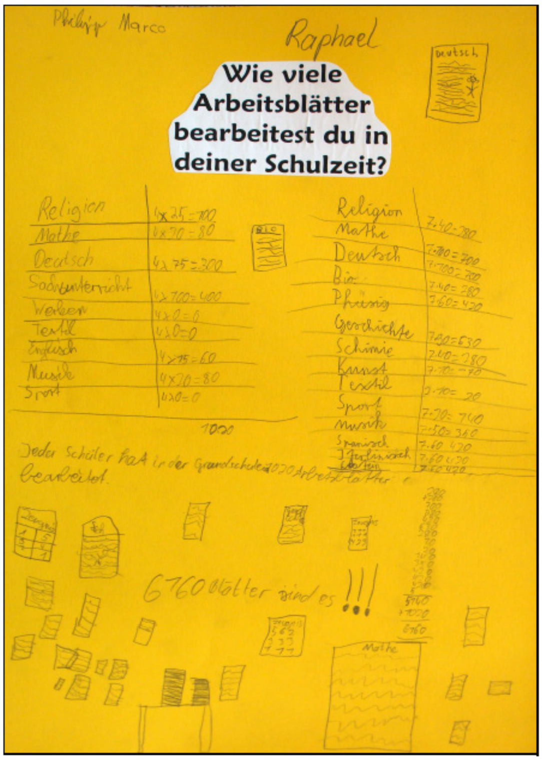

Figure 5: Poster designed by Team B1

For a more detailed assessment of the pupils' work progress, a more process oriented analysis could be valuable. The refined category system developed for this purpose will be presented in the following section.

Category system

Some of the categories which distinguish the work processes have already been mentioned in previous sections of this article. Overall, the observed (videotaped and transcribed) work processes can be described by a category system consisting of six "basic categories" of modelling, and further six categories including "interaction" as well as "further activities". For a more detailed description, we can define additional six sub-categories of modelling which further specify the basic categories.[7] It thus seems appropriate to differentiate between estimations, discussions and research within the basic category "collecting data", and between calculations, argumentations and control within "processing data", for example. The suggested differentiation of basic- and sub-categories enables, on the one hand, a very precise description of the observed processes. On the other hand, the variable degree of resolution also makes the category system applicable to other Fermi questions or modelling tasks.

[7] It goes without saying that only certain combinations are useful.

The following overview names the categories, briefly explains them and, for some cases, gives additional illustrative examples.

Basic categories of modelling	*Sub-categories of modelling*	
Orientation – understanding the problem	Spontaneous solution	S
Orientation – first ideas	Calculations	C
Planning	Estimations	E
Collecting data	Argumentation / Discussion	D
Processing data	Control	O
Reflection	Use of auxiliary material / Research	M
Teacher-student interaction	*Further activities*	
Report to external party / Ask for help	Team work / Preparing poster / Secure data	
Impulse	Context	
	No reference to the problem	
	Others	

Basic categories of modelling

- Orientation – understanding the problem: It is about the basic understanding of the problem.

Number	Person	Statement	Category	Sub-category
3	P	How many work sheets do you complete during your time at school? ** No idea.		
4	M	Ohh that will be hard. [LAUGHING]		

- Orientation – first ideas: First ideas on how to solve the problem are expressed. However, these do not lead far and must ultimately be abandoned or modified.

8	M	### we should check up every ring binder,		
		…		
14	R	… then we have round about 1000'		S

- Planning: A critical idea for solving the problem is found, or further proceedings are planned with foresight. Possibly, metacognitive approaches can be identified.

43	M	I've got an idea: we can write down the school subjects,		

- Collecting data: Data is collected, usually by applying everyday knowledge or making estimations, also by using auxiliary means or research.

53	M	In mathematics we do not have so many,		D
54	R	Sure,		D
55	P	Nope,		D

- Processing data: Collected data is processed. This takes place mainly by carrying out calculations. Justifying a certain calculation is also a constitutive part of data processing.

151	R	75 times 4 is erm*		C
152		*5sec*		C
153	P	2 times 75-		C
154	R	300,** isn't it'		C

- Reflection: Pupils mainly reflects on the meaning of the results, reflection on the approach can also take place (while this must not be confused with control of individual calculations).

555	R	Man, a great many! 6160, if one thinks about,		
556	P	Hm,		
557	R	Imagine you have to copy so many -		
558	P	hohoho [LAUGHING]		
559	R	How expensive would it be'		

Sub-categories of modelling

- Spontaneous solutions: Spontaneous solutions are very rough estimations of the solution or simply an imagined result that cannot be further explained.
- Calculations: Calculations are subordinated to data processing. Very simple auxiliary calculations and conversions of units are assigned to this category, just like complicated calculations of the final result.
- Estimations: Estimated values are allocated to this category as well as their possible justification, as latter usually refers to basic supporting knowledge which is linked to the estimations.

482	R	7 times 20* 140, music' because of all the songs,		E
483	M	30' oh no, more,		E
484	P	40 round about,		E
485	R	I'll write down 50,		E

- Argumentation / Discussion: Processes of argumentation and discussion take place during many phases of the modelling process. In categorisation, they are highlighted only in processing and collecting data. In this context, data processing mainly implies discussing possible approaches; in data collection, it is above all about justifying the selection of data.
- Control: With regard to control, we also differentiate between data collecting and data processing. In former case, it is rather the completeness of selected data that is checked, in latter case the individual calculations.

285	R	Then only 3,		O
286	P	Only 3, let's take Spanish-		O
287	R	Latin and Italien,		O

- Use of auxiliary material / Research: This can be understood as a sub-category of collecting data. Auxiliary means or tools vary according to the problem at hand. When conducting research, both, books as well as the Internet or other sources can be drawn upon.

Teacher-student interaction

- Impulse: Impulses come from the teacher. During the time that her students are working on the problem, the teacher acts above all as a supervisor who gives support in case of questions or problems. Her comments and remarks that were made in these cases are called impulses.
- Report to external party / Ask for help: Students come into interaction with the teacher. They report on their current state of progress or search for assistance by asking questions.

Further activities

- Team work / Prepare poster / Secure data: Processes in this category include all group work, the making of the poster and data backup.
- Context: This category points to statements made that lie within the context of the problem situation.
- No reference to the problem: This category summarises all statements made by students which bear no reference to the problem itself.
- Others: A rest category includes all statements that are not categorisable, or events which cause an interruption of the process through external circumstances, such as a break.

We allocated Möwes-Butschko's category "Data securing" to the category "Team work / Preparing poster / Secure data"; this may be due to the nature of the assignment given to the pupils, who used their poster for all notes, calculations etc. from the beginning on. Phases of review (cf. Pólya's model) or metacognition could not be clearly identified – also because the remarks in question can be subject to a wide scope of interpretation. Therefore, they were not explicitly included in the category system.

Analysis of the work processes

Within the limits of this paper we can thus only describe work processes with a low degree of resolution, i. e. we must neglect the sub-categories. If you determine the relative amount of time Team B1 spent on the respective categories based on the pupils' verbalizations – which makes a certain "blurring" unavoidable – the result is the following diagram.

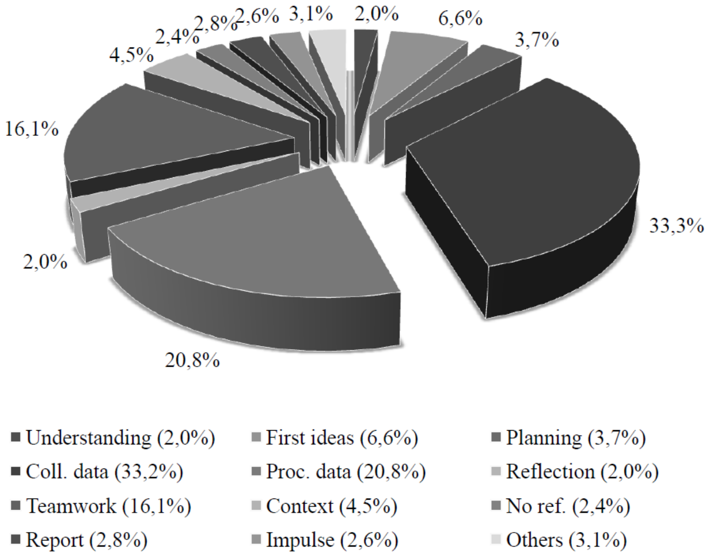

Figure 6: Percentage of time spent on each process category (Team B1)

This diagram confirms the impression we got during the observations: The pupils worked on the data very intensively, data collection and processing took a lot of time. It also becomes clear that, in general, the pupils handled the questions very concentrated and quite independently.

The progress of problem solving is pictured in the following graph. Every field represents one second and every row one minute; it should thus be read line by line from left to right. Categories are indicated by respective colour shades and hatchings.

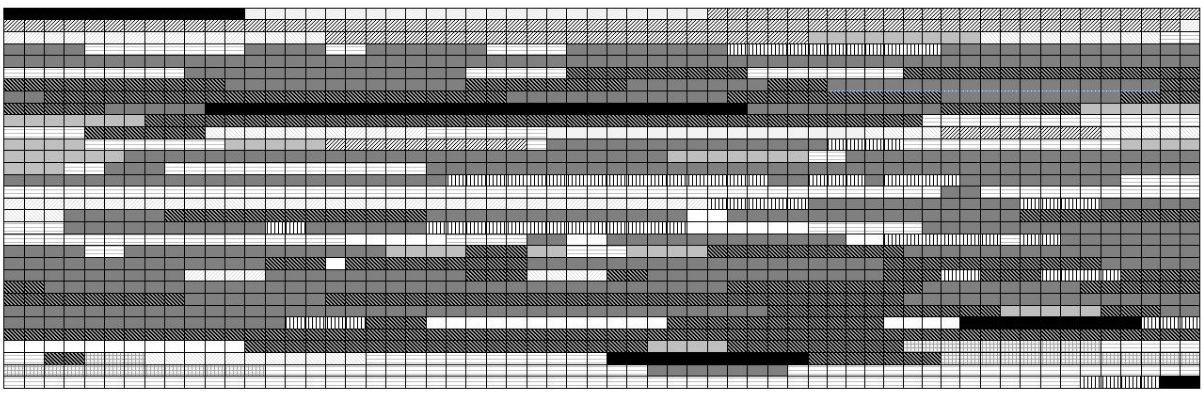

Figure 7: Progress of work processes (Team B1)

Obvious is a longer phase of orientation right at the beginning, during which pupils first of all tried to develop their own understanding of the question and developed first ideas. Towards the end of the process, a focus naturally laid on designing the poster. Before that, we could observe a phase of reflection, which "only" focused on the results, however, and not on one's own work processes that had previously been carried out.

Apart from other details, the graph clearly illustrates that even work processes described by basic categories of modelling can *be repeated* and *without following a strict order* – for example, phases of orientation and planning took place again during a later stage of the progress.

The second team of pupils was working on the question of how much time of your life you spend brushing your teeth. The answer they came up with was 35 days. This relatively short time span can be explained by the fact that, while the pupils did discuss how often you brush your teeth per day, they made their final calculations taking into account only 2 minutes of brushing per day.

The relative amount of time spent on the respective categories is shown in Figure 8.

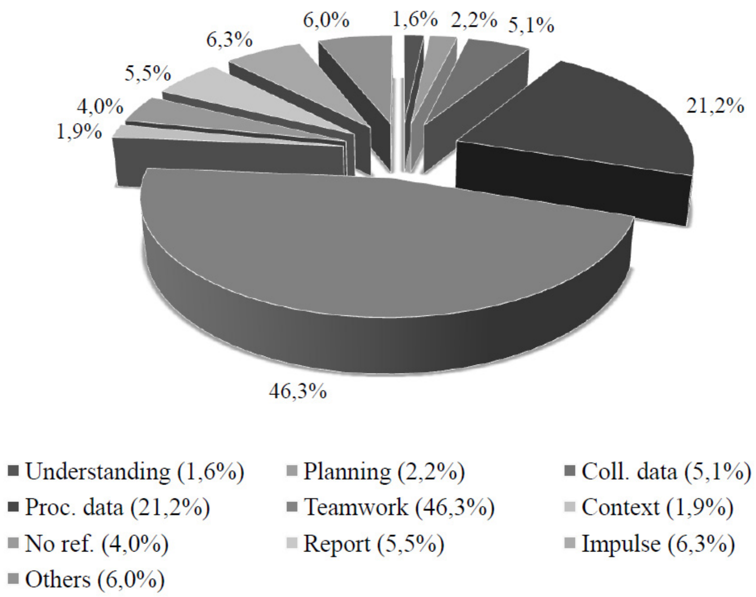

Figure 8: Percentage of time spent on each process category (Team B2)

As was to be expected, data collection was much less time consuming due to the nature of the question. As the required calculations are more difficult (different time units), the relative amount of time spent on data processing is even a little higher than with the first group. We could not observe a reflection of the results or of the process. Striking is the large percentage of the category “Team work / Preparing Poster / Secure data“.

The following graph shows the progress of work processes.

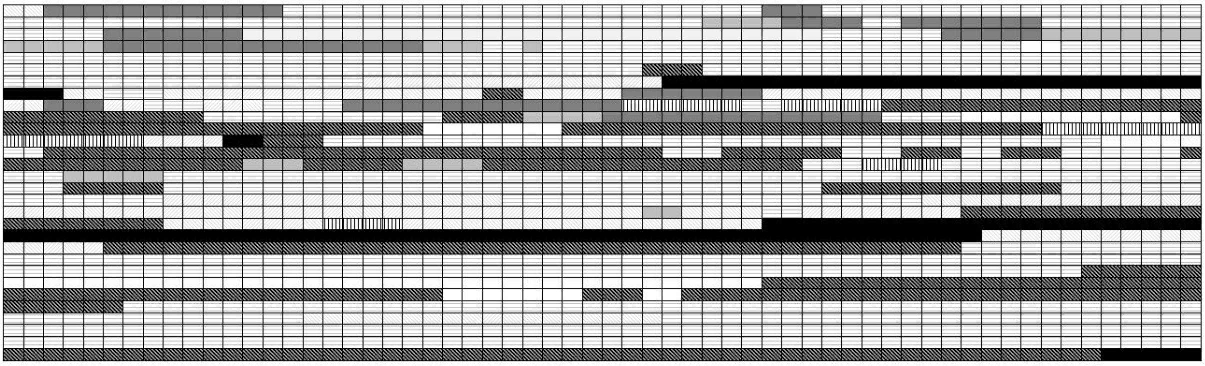

Figure 9: Progress of work processes (Team B2)

The graph suggests that it took the pupils a while in the beginning to initiate cooperation within the group, and also at a later stage, coordinating cooperation and poster design took up a lot of time. The group was quite obviously dependent on repeated impulses and assistance by the teacher.

6. Discussion

Also due to the observations made in the other groups, as well as in the two classes on the whole, using Fermi questions in class towards the end of primary school (i. e. Year 4) generally seems possible to us. They can encourage persistent involvement, initiate important work processes that go beyond the respective question at hand – especially regarding data handling – and simultaneously offer reasonable exercises for basic mathematical skills which are regarded as useful from a pupil's point of view. Metacognitive elements, however, could not be observed among the groups that took part in this study.

An important aspect from our point of view when using Fermi questions – as Team B2's work progress also suggests –, is that you should not distract too much from content-related aspects by focusing too much on the methodological design of the lesson.[8]

The constructed category system derived from content analyses combines aspects of mathematising non-mathematical questions or situations, and aspects of problem solving.

To us, it seems appropriate to describe model building processes among primary school pupils in more detail, also in their progress. This way, it is e. g. possible to trace varying demands of Fermi questions on pupils, or an according handling of these questions (e. g. extent of collecting data).

[8] With regard to the aim of this study, we find it nonetheless appropriate to e. g. ask the pupils to design a poster which also shows considerations and interim results. This way, it is easier to understand the approaches and steps that pupils take to find a solution to the problem.

Due to the fact that this study was based on case studies, possible correlations between the progress of model building processes or the percentage of time spent on certain process categories, and the final success of the process, could not be explored. This could be an aim of a more extensive study in the future.

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J. Kürschák a world-famous scholar teacher (1864-1933)

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Abstract

In this paper we shall commemorate J. Kürschák on the occasion of the 145-th anniversary of his birth. We will present his life and work, his world-famous mathematical results as the theory of valuations, geometric constructions with "Aichmass", his miniatures as the area of a regular dodecagon inscribed in a circle, the problem posed on the 1977 International Mathematical Olympiad (Beograd), Kürschák's tiling, combinations with repetition, the knight's tour on the infinite chess-board. At last we will mention his talent developing work. We discuss why the most prestigious Hungarian mathematical contest is named after him and the book Problems of Mathematics Contest (1929) collected and annotated by him.

Keywords

commemoration on J. Kürschák, Kürschák's theorem, Kürschák's tiling, problem solving, Kürschák contests, Hungarian Problem Book.

Introduction

J. Kürschák, the outstanding Hungarian mathematician, was born 145 years ago. He was a world famous scholar teacher, mathematical professor, mathematician, member of the Hungarian Academy of Science.

His name is well-known among high school students, a lot of them have heard about the Kürschák competitions, Kürschák's problems, or they read on the website about Kürschák's tiling.

In Hungary his name is less mentioned than the name of his contemporaries: *F. Riesz* or *L. Fejér*. He was the *"grey eminence"* of the mathematical life of the 20-th century.

His most famous students were: *J. v. Neumann* and *D. Kőnig*. The latter wrote that J. Kürschák serves as an ideal to the Hungarian youth.

J. Kürschák

Kürschák's life and work

About Kürschák's life

He was born on 14 March 1864 in Buda(pest) and died on 26 March 1933 in Budapest. He was son of a craftsman, who died when Kürschák was 6 years old. He, with his brother, was brought up by their mother. He attended secondary school with sciences (Realschule) in Buda. His mathematics teacher - L. Kreybig - advised him to continue his studies at the Technical University of Budapest. He graduated in 1886 as a secondary school teacher of mathematics and physics. At the University he was influenced by the lectures of *J. Hunyadi* and by the seminars of *J. Kőnig*. His pedagogical ideal was J. Kőnig. From him had learnt Kürschák how to stimulate students to work independent. After finishing university studies he taught at secondary schools (Bratislava, Debrecen, Roznava). He was secondary school teacher for 6 years, but in his whole life he did a lot for the talented pupils and for the teachers. He gave courses for future high school teachers mainly on elementary geometry and geometrical constructions. Several of his papers deal with the teaching and popularization of mathematics. His devotion to intelligently guided problem solving is illustrated by his famous problem book the so called Hungarian Problem Book.

Kürschák wrote his first article as an undergraduate student: *Über Kreis ein- und umgeschriebene Vielecke* (1887). In this article he investigated extremal properties of polygons inscribed and circumscribed about a circle.

Nowadays we can read his article series on the *History and theory of circle measuring* (I-IX. Math. Phys. Lapok 1892-1894) on the website of Wikipedia.

In his whole life, from 1888 until 1933, he taught in Budapest at the Technical University. He received his doctorate in 1890, he made his "habilitation" in 1891, he became corresponding (1896) and later (1914) full member of the Hungarian Academy of Sciences. He taught analysis, algebra, geometry, differential geometry. He led special seminars for the teacher, architecture and chemist students.

He liked independent work. His opinion was: *"Really I understood from science as much as I independently thought over or I promoted a little."*

Kürschák published all of his important articles in two languages: Hungarian and German. The German publications appeared in famous Journals (Math Annalen, Crelle Jornal), the number of his publications is more than 100. The themes of his investigations were: geometry, differential geometry, algebra and number theory, matrices and determinants, analysis, history of mathematics. He wrote articles in the Pallas Lexicon, lecture notes and the famous Hungarian Problem Book too. He took part in the preparing of *the second edition of W. Bo-*

lyai's Tentamen, which contains the Appendix, the excellent work of János Bolyai about non-euclidean geometry.

About Kürschák's work

We will deal with Kürschák's most important works:

I. Outstanding mathematical results

1. Theory of valuations
2. Geometric constructions with "Aichmass"

II. Miniatures

1. The area of a regular dodecagon inscribed in a circle and the problem posed on the 1977 International Mathematical Olympiad (Beograd), Kürschák's tiling
2. Combinations with repetition
3. The knight's tour on the infinite chess-board

III. Talent developing work (Hungarian mathematical contests named after Kürschák, the Hungarian Problem Book (1929) collected and annotated by him).

I. Outstanding mathematical results

1. Theory of valuations

Kürschák's main achievement is the founding of the theory of valuations. All books of modern algebra mention the name of J. Kürschák as the creator of one new part of the modern algebra in the 20-th century, the theory of valuations. He succeeded in generalizing the concept of absolute value. His method was developed later by *A. Ostrowski* into a consistent and very important arithmetical theory of fields.

Valuation is a function on a field that provides a measure of size or multiplicity of elements of the field. It generalizes to commutative algebra the notion of size inherent in consideration of degree of a pole or multiplicity of a zero in complex analysis, the degree divisibility of a number by a prime number in number theory, and the geometrical concept of contact between two algebraic or analytic varieties in algebraic geometry.

If K is an arbitrary (commutative) field then valuations are $|\ \ |: K \to R$ functions, with the properties:

- $|a| \geq 0, (a \in K)$
- $|a| = 0 \Leftrightarrow a = 0\ (a \in K)$
- $|a\, b| = |a|\, |b|\ (a, b \in K)$
- $|a + b| \leq |a| + |b|\ (a, b \in K).$

A field is valuated if it exists a valuation on it.

Kürschák presented his most important scientific result at first Hungarian, and then at the *International Mathematical Congress in Cambridge (1912)*. In 1913 appeared his article *Über Limesbildung und allgemeine Körpertheorie*. *H. Hasse*'s divisor theory is based on Kürschák's results (*Zahlentheorie, 1949*). *J. Neukirch* (1939-1979) in his monograph *Algebraische Zahlentheorie (1992)* mentions that J. Kürschák laid the foundation of p-adic valuation. In 1922 *V. D. Gohkale* in his article *Concerning Compact Kürschák's fields* (American J. of Math. Vol. 44) introduced the definition of Kürschák's field: "Kürschák field is a field with modulus".

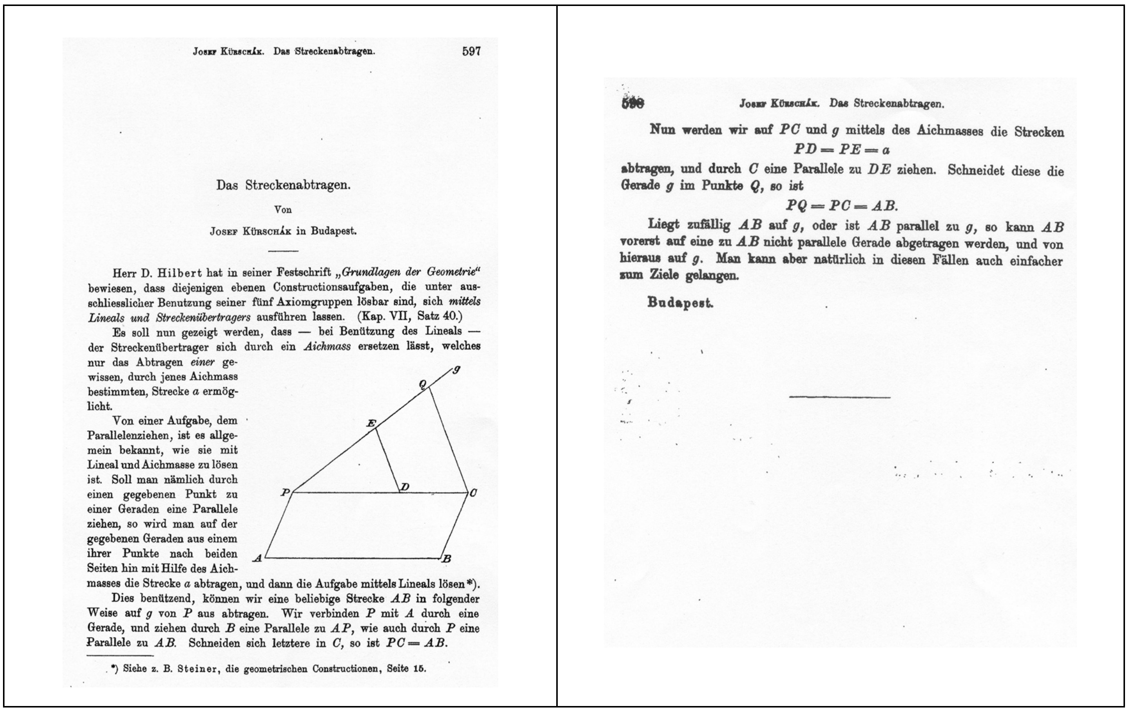
Kürschák wrote with *Hadamard* an article about the number fields in the *Encyclopédie des Sciences*. This theme is analysed by *J. Gray: König, Hadamard and Kürschák and abstract algebra* (Mathematical Intelligencer 1997. Vol.19. No. 2.)

2. Geometric constructions with "Aichmass"

Kürschák's article *Das Streckenabtragen* (Math. Ann. 1902) is very fundamental and very short, only 1 ½ pages long.
Kürschák completes a part of *D. Hilbert's work Foundation of Geometry* (Grundlagen der Geometrie, 1899). From the second edition of Hilbert's "Bible of Geometers" we can find Kürschák's result, with his name, with the original text and figures of Kürschák.

In the theory of construction Kürschák demonstrated that a single compass of fixed span (unit transfer: Aichmass) could be a substitute for a compass of variable span. He has shown the sufficiency of ruler and of a fixed distance for all discrete geometrical constructions.

Problem:
There is given a straight line *g* and the segment *AB*. (*AB* is not parallel to *g*). We have to transfer segment *AB* with the help of a single compass of fixed span on the straight line *g*.
We will present Kürschák's original solution (Fig.1). It is very elegant. We can present it to secondary school students.



JOSEF KÜRSCHÁK. Das Streckenabtragen. 597

Das Streckenabtragen.

Von

JOSEF KÜRSCHÁK in Budapest.

Herr D. Hilbert hat in seiner Festschrift „*Grundlagen der Geometrie*“ bewiesen, dass diejenigen ebenen Constructionsaufgaben, die unter ausschliesslicher Benutzung seiner fünf Axiomgruppen lösbar sind, sich *mittels Lineals und Streckenübertragers* ausführen lassen. (Kap. VII, Satz 40.)

Es soll nun gezeigt werden, dass — bei Benützung des Lineals — der Streckenübertrager sich durch ein *Aichmass* ersetzen lässt, welches nur das Abtragen *einer* gewissen, durch jenes Aichmass bestimmten, Strecke a ermöglicht.

Von einer Aufgabe, dem Parallelenziehen, ist es allgemein bekannt, wie sie mit Lineal und Aichmasse zu lösen ist. Soll man nämlich durch einen gegebenen Punkt zu einer Geraden eine Parallele ziehen, so wird man auf der gegebenen Geraden aus einem ihrer Punkte nach beiden Seiten hin mit Hilfe des Aichmasses die Strecke a abtragen, und dann die Aufgabe mittels Lineals lösen*).

Dies benützend, können wir eine beliebige Strecke AB in folgender Weise auf g von P aus abtragen. Wir verbinden P mit A durch eine Gerade, und ziehen durch B eine Parallele zu AP, wie auch durch P eine Parallele zu AB. Schneiden sich letztere in C, so ist $PC = AB$.

*) Siehe z. B. Steiner, die geometrischen Constructionen, Seite 15.

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Nun werden wir auf PC und g mittels des Aichmasses die Strecken

$$PD = PE = a$$

abtragen, und durch C eine Parallele zu DE ziehen. Schneidet diese die Gerade g im Punkte Q, so ist

$$PQ = PC = AB.$$

Liegt zufällig AB auf g, oder ist AB parallel zu g, so kann AB vorerst auf eine zu AB nicht parallele Gerade abgetragen werden, und von hieraus auf g. Man kann aber natürlich in diesen Fällen auch einfacher zum Ziele gelangen.

Budapest.

Figure 1

II. Miniatures

1. The area of a regular dodecagon inscribed in a circle

Most well-known and quoted miniature of Kürschák is the so called *Kürschák's theorem* about the *area of a regular dodecagon inscribed in a circle* (Math. Phys Lapok, 1898). This theorem is the base of Kürschák's tiling too.

Only the square and the regular dodecagon, among the regular polygons inscribed in a circle, have rational areas.

Kürschák's theorem

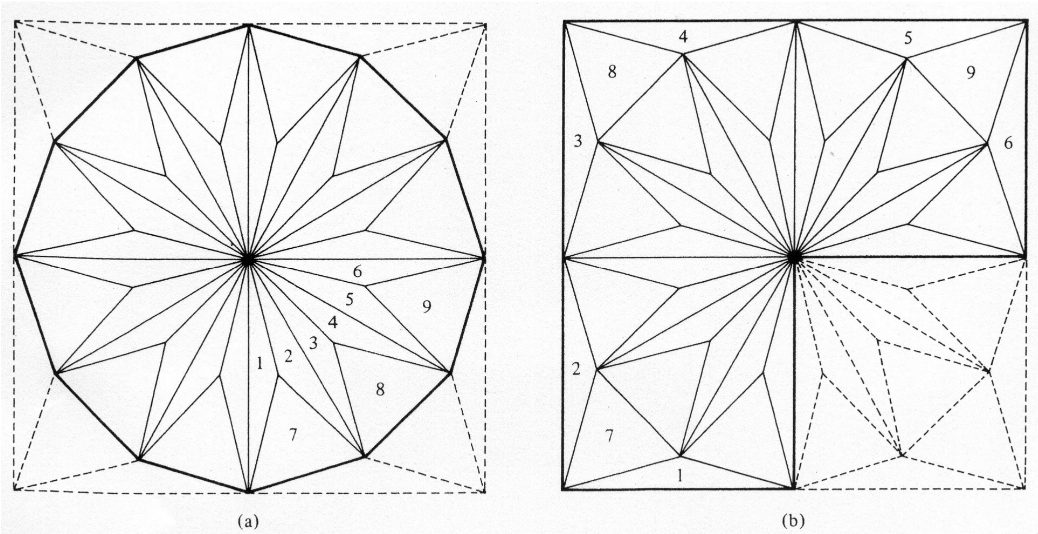
The area of a regular dodecagon inscribed in a circle with radius R is $3R^2$.

Proofs:

1. way: Nowadays a student uses trigonometry to prove this theorem.

$$T = 6R^2 \cos\frac{\pi}{6} = 3R^2.$$

2. way: Kürschák choose another method, the *method of W. Bolyai*, *the method of decomposition.* His original figure we can see on the Figure 3. The animations use this method too.



Figures 2 (a) and (b)

This geometric way can seen on the Figures 2 *a* and *b*. We hope that this visual proof can be used to enrich students work.

It is an interesting and worthwhile problem for the students to prove that the square circumscribed about the regular dodecagon is made up of two sets of congruent triangles. The dodecagon is inscribed in a circle with radius *R*, so the square has side *2R*.

They have to verify that there are *16* equilateral triangles and *32* isosceles triangles with angles 15^o - 15^o – 150^o and the longest side *R*. The area of the large square is $4R^2$, and it consists of *4* smaller squares each of area R^2. They take the numbered triangles inside the dodecagon in this square, and move them to cover the triangles with like numbers in the other three squares, completely to cover all three squares. Now we find that the total area is $3\ R^2$, so we got the area of the original dodecagon (Figure 4).

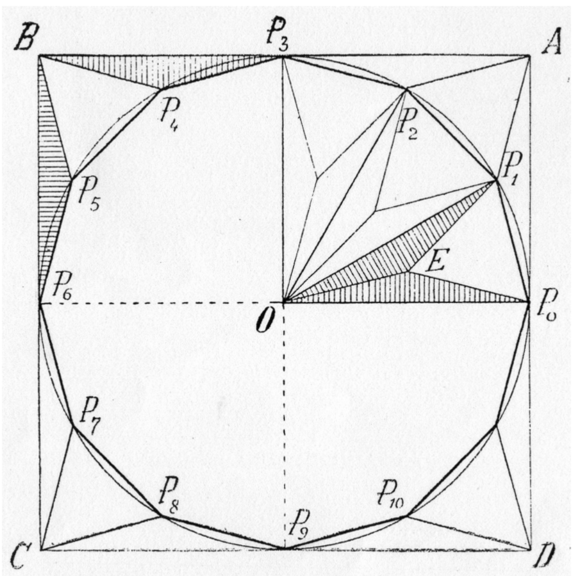


Figure 3

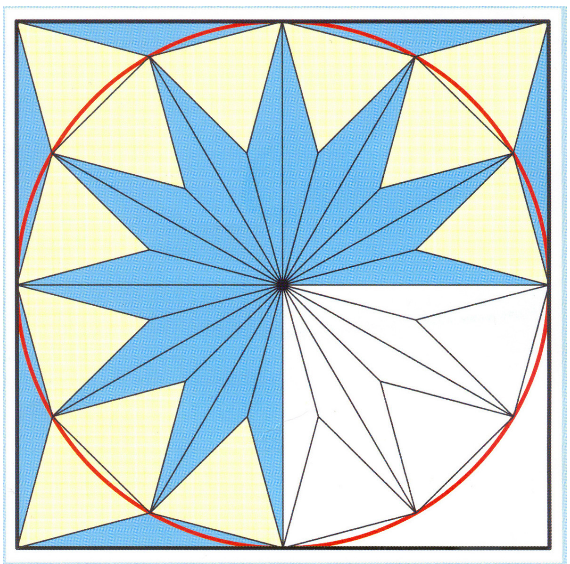


Figure 4

Such a way exists since two polygons which have equal area are equidecomposable in consequence of the *Bolyai-Gerwin theorem.*

The tiling of the square is known as *Kürschák's tile.* There are other Kürschák's tiles too. (Figure 5)

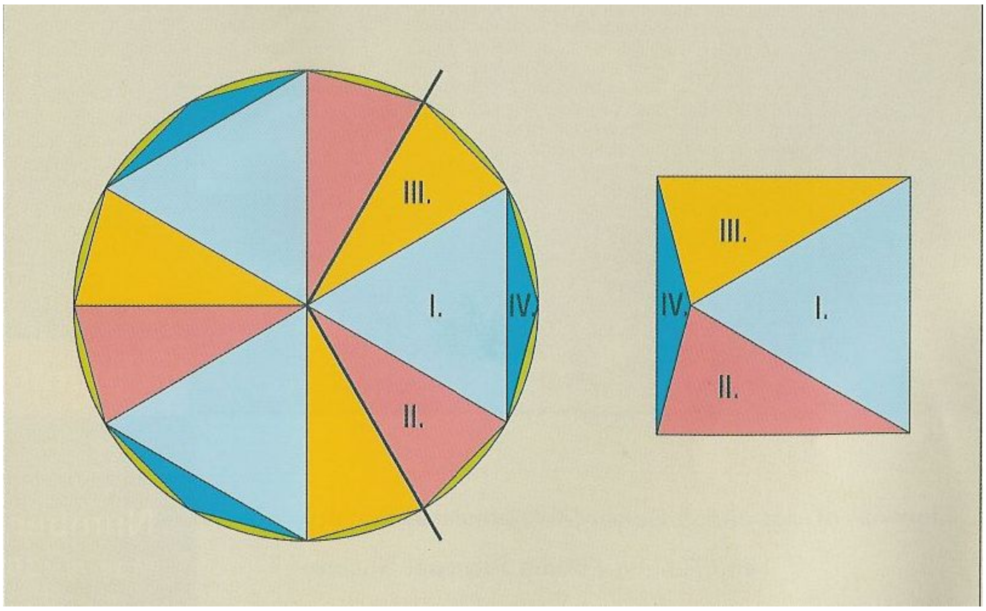


Figure 5

A version of Kürschák's theorem was the *first problem at the 1977 International Mathematical Olympiad (Figure 6)*

Problem 1977/1 IMO (Beograd)

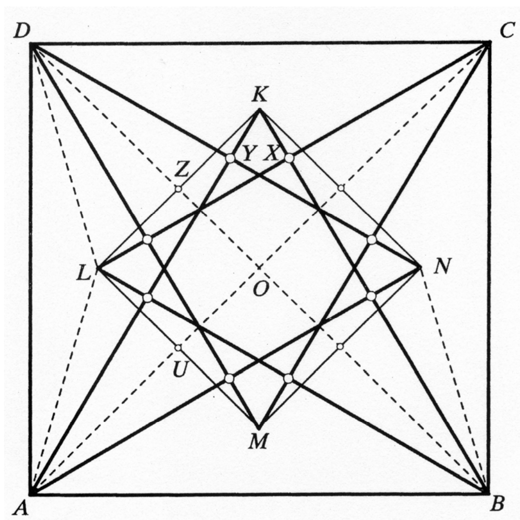


Figure 6

Equilateral triangles *ABK, BKL, CDM, DAN* are constructed inside a square *ABCD.* Prove that the midpoints of the four segments *KL, LM, MN, NK* and the midpoints of the eight segments *AK, BK, BL, CL, CM, DM, DN, AN* are the twelve vertices of a regular dodecagon (Figure 6).

It is not too difficult to establish the angle measures needed to demonstrate that *KLMN* is a Kürschák's tile, and this in turn demonstrates that the polygon is a regular dodecagon.

2. Combinations with repetition *(Math. Phys Lapok, 1925, 7-8)* (Figure7)

In this miniature Kürschák gives a descriptive proof for the calculation the number of combinations with repetition. This article has a Hungarian and also a German version. It is obvious that we could present it to the high school students. It is a very simple method which he advises for introducing concept of the combination with repetition.

ÜBER DIE KOMBINATIONEN.

Die Anzahl der Kombinationen k-ter Klasse *mit Wiederholungen* von n Elementen ist gleich der Anzahl der Kombinationen k-ter Klasse *ohne Wiederholungen* von $n+k-1$ Elementen. Für diesen bekannten Satz wird hier ein anschaulicher Beweis gegeben, der im wesentlichen eine Versinnlichung des abstrakteren Beweises von SCHERK ist [Crelle Journal, Bd. 3, (1828), S. 96—97.] — Es wird von den folgenden, aus k Reihen und $n+k-1$ Spalten bestehendem Schema ausgegangen:

$$\begin{array}{l} 1\ 2\ 3\ \ldots\ n \\ \quad 1\ 2\ 3\ \ldots\ n \\ \qquad \ldots\ldots\ldots\ n \\ \qquad\quad 1\ 2\ 3\ \ldots\ n. \end{array} \qquad (T)$$

In jeder Kombination $\alpha_1, \alpha_2, \ldots \alpha_i, \ldots, \alpha_k$ m. W. von $1, 2, \ldots, n$ seien die Elemente in ihrer Größenfolge angeordnet; γ_i besage dann, in der wievielsten Spalte von (T) die i-te Zahl α_i einer Kombination aufzufinden ist, wenn wir sie *in der i-ten Zeile* suchen. Der Satz folgt nun daraus, daß auf diese Weise jeder Kombination $\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_k$ m. W. von $1, 2, \ldots, n$ eine (nach der Grösse der Elemente geordnete) Kombination $\gamma_1, \gamma_2, \ldots, \gamma_i, \ldots, \gamma_k$ o. W. von $1, 2, \ldots, n+k-1$ entspricht und umgekehrt.

Josef Kürschák.

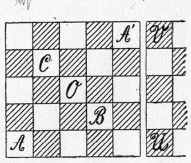
Figure 7

3. The knight's tour on the infinite chess-board (*Acta Math. Szeged. 1928, 12*-13, Figure 8)

LÓUGRÁS A VÉGTELEN SAKKTÁBLÁN.

Célom bebizonyitani, hogy *a végtelen sakktábla úgy futható be lóugrásoknak (mindkét értelemben határtalan) sorozatával, hogy minden mezőre egyszer és csak egyszer érünk.*

1. Az 5^2 mezős véges tábla befutása. Jelentse (1. ábra) O valamely 5^2 mezős véges táblán a középső mezőt, A pedig



1. ábra.

7	12	19	24	5
20	**25**	6	13	18
11	8	17	4	23
16	21	2	9	14
1	10	15	22	3

2. ábra.

valamelyik sarok-mezőt; B és C legyenek az A-t nem tartalmazó diagonálisnak O-val érintkező mezői.

A-ból akár B-be, akár C-be az 5^2 mezős táblán lóugrásokkal úgy juthatunk, hogy minden mezőre éppen egyszer érünk.

A-ból C-be a kívánt módon úgy juthatunk, hogy a mezőket a 2. ábrában látható számozás rendjében futjuk be.*

* Az 5×5 mezős tábla számos lóugrásos befutását tartalmazza EULER híres értekezése: *Sulution d'une question curieuse qui ne paroit soumise à aucune analyse,* Histoire de l'Académie, Berlin 15. köt., (bemutatva 1756., kinyomatva 1761., a címlapon hibásan 1766-os évszámmal), 310—337. lap. (L. különösen a 36—40. §-t a 332—335. lapon.) A fenti 2. ábra a 335 lap második ábrájától csak az 5.—25. mező fordított számozásában különbözik.

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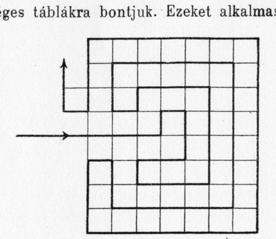
Ha A-ból B-be akarunk jutni, csak a sorok és oszlopok szerepét kell felcserélnünk.

2. Átlépés 5^2 mezős tábláról vele szomszédosra. *Két egymás mellé tett 5^2 mezős tábla közül bármelyik, ennek bármely sarok-mezejéből kiindulva, úgy futható be lóugrásokkal, hogy az utoljára elért mezőről a másik táblának legközelebb eső sarok-mezejére lóugrással juthatunk.*

Valóban az 1. ábrában a bal táblán akár A-ból, akár A'-ből eljuthatunk B-be és innen lóugrással a másik tábla U mezejére.

Hasonló módon juthatunk a bal tábla másik két sarok-mezejéből V-be.

3. A végtelen tábla befutása. A végtelen sakktáblát 5^2 mezős véges táblákra bontjuk. Ezeket alkalmas sorrendben



3. ábra.

úgy futjuk be, hogy mindegyikre sarok-mezőn lépünk s onnan úgy járjuk be lóugrásokkal, hogy azután a következő táblának valamelyik sarok-mezejére ugorhassunk.

Hogy az 5^2 mezős táblákat hogyan fűzhetjük egy sorozatba, az a 3. ábrán látható. Rajta minden rácspont egy-egy 5^2 mezős táblát képvisel. Hogy e táblákat milyen egymásutánban futjuk be, azt a végtelenből benyúló vízszintes és ennek kanyargó folytatása mutatja.

Kürschák József.

Figure 8

We know that Euler dealt with the problem of knights on the chessboard (1756). Kürschák in his article shows at first how is possible to ramble all over the finite 5x5 chessboard with knight tours and then discusses the infinite case.
This article is connected with the first problem of the Eötvös Competion of 1926.

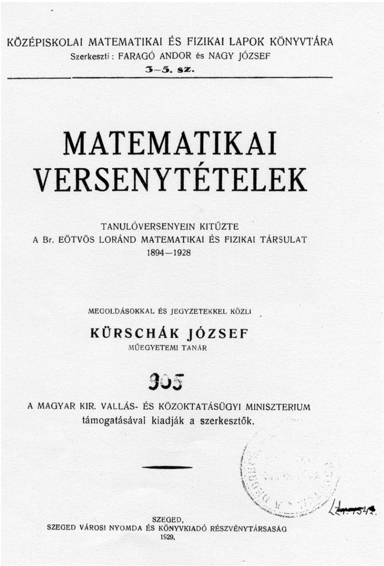
***Problem* 1 (Eötvös Competition, 1926)**
Prove that, if a and b are given integers, the system of equations
$x+ y + 2z + 2t = a$
$2x - 2y + z - t = b$ has a solution in integers x, y, z ,t.

Kürschák gave a ***Note*** to its solution: *Moves of the knight on an infinite chessboard.* The theorem proved in this problem is equivalent to the statement:
On an infinite chessboard, any square can be reached by the knight in a sequence of appropriate moves. An infinite chessboard differs from the usual one that it extends over the entire plane.

III. Talent developing work

Kürschák was a versatile and thought provoking teacher. He was one of the main organizers of mathematical contests. He contributed to the selection and education of many brilliant students. He played the main role in developing the mathematical ground for scientific talent.
The *Eötvös Competition* has started for Hungarian high school students in 1894. It was organized by the Hungarian Mathematical and Physical Society. From the same time appeared the Mathematical Journal for Secondary School Students (Középiskolai Matematikai Lapok) too. After the Second World War the Eötvös Competition was named after Kürschák. The *Kürschák Competition* is the most prestigious Hungarian mathematical competition nowadays. Among the winners of the competition many turned into scientist of international fame: *L. Fejér, A. Haar, M. Riesz, T. Kármán, D. Kőnig, G. Szegő, A. Kóródi, T. Radó, L. Rédei, L. Kalmár, E. Teller, L. Tisza, T. Gallai, T. Szele, Á. Császár, L. Lovász*, etc.
Kürschák took part in the organization and evaluation of the Eötvös Contests, often posed problems. He collected the posed problems (1894-1928), annotated the solutions of the students combining his excellence in mathematics with his interest in education when he supplied the elegant solutions and illuminating explanations.
This book *Problems of Mathematics Contests* (Figure 9) was published at first by Kürschák in 1929, later it was followed by completed new editions. The editors were: *Gy. Hajós, Gy. Neu-*

komm and *J. Surányi* (1956), *Gy. Hajós*, and *J. Surányi* (1964), *J. Surányi* (1986, 1992). It was translated on different languages (Japan, Russian, Rumanian, etc.)


KÖZÉPISKOLAI MATEMATIKAI ÉS FIZIKAI LAPOK KÖNYVTÁRA
Szerkeszti: FARAGÓ ANDOR és NAGY JÓZSEF
3–5. sz.

MATEMATIKAI VERSENYTÉTELEK

TANULÓVERSENYEIN KITŰZTE
A Br. EÖTVÖS LORÁND MATEMATIKAI ÉS FIZIKAI TÁRSULAT
1894—1928

MEGOLDÁSOKKAL ÉS JEGYZETEKKEL KÖZLI
KÜRSCHÁK JÓZSEF
MŰEGYETEMI TANÁR

A MAGYAR KIR. VALLÁS- ÉS KÖZOKTATÁSÜGYI MINISZTERIUM
támogatásával kiadják a szerkesztők.

SZEGED,
SZEGED VÁROSI NYOMDA ÉS KÖNYVKIADÓ RÉSZVÉNYTÁRSASÁG
1929.

Figure 9

Figure 10

The Random House in 1961 published on the base of the second Hungarian edition the *Hungarian Problem Book I-II* (Figure 10*)* with the Preface of Professor G. Szegő, one of the former contestants. He wrote:

"*The problems are almost all from high school material (no calculus included), they are of an elementary character, but rather difficult, and their solution requires a certain degree of insight and creative ability.*

Mathematics is a human activity almost as diverse as the human mind itself. Therefore it seems impossible to design absolutely certain and effective means and methods for the stimulation of mathematics on a large scale. The competitive idea seems to be powerful stimulant."

Later was published the *Hungarian Problem Book III.*

The *Hungarian Problem Books* are the best books of the world for preparing students to the mathematical competitions. It is believed that it helps the Hungarian contestants to be good at the International Mathematical Competitions. We present the second problem from the 1913 Competition.

***Problem 2* (Eötvös Competition, 1913)**

Let *O* and *O'* designate two diagonally opposite vertices of a cube. Bisect those edges of a cube that contain neither of the points *O* and *O'*. Prove that these midpoints of edges lie in a plane and form the vertices of a regular hexagon.

The winner of the contest was *Tibor Radó*. Nowadays we can find this problem in opened form in our schoolbooks.

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Primary teacher students' strategies in comparing the cardinalities of infinite sets

Dr Vida Manfreda Kolar

Dr Tatjana Hodnik Čadež

Abstract

The article discusses the development of the concept of infinity in children. It focuses on a number of difficulties that children cope with when dealing with problems related to infinity such as its abstract nature, understanding of infinity as an ongoing process which never ends, understanding of infinity as a set of an infinite number of elements, and understanding of well known paradoxes. Special attention is given to the problem of comparing infinite sets. The results of some existing studies which show that the perception of infinity depends on the context are presented.

In the empirical section of the article, a study is described that was conducted with primary teacher students with the aim of researching their understanding of the concept of infinity. The focus was on finding out how students who received no in-depth instruction on abstract mathematical content, such as Cantor's set theory, tackle problems related to the comparison of infinite sets, and what argumentation they provide for their answers.

The results show that some of the respondents possess a degree of intuition about the solution to the problem but they seem to lack the mathematical knowledge required to provide proper argumentations for their assumptions. Their explanations are poor, none of them choose to use the method of one to one correspondence (creating pairs of elements), which means that this method is not a part of their intuition when dealing with the problem of comparing infinite sets.

Key words

infinity, concept of infinity, infinite sets, pairwise correspondence, primary teacher students

The development of the concept of infinity in children

Conceptualizing the infinite is an attractive subject of conversation for even the youngest inquisitive minds in the preschool period. Piaget and Inhelder (1956) conducted one of the first studies of children's understanding of infinity. Their study involved geometrical problems such as how to draw the smallest and the largest possible square on a piece of paper, or what would happen if the process of division of a geometrical figure (for instance by two) were to be continued mentally. What would be the form of the final element of such a division? They concluded that in the concrete operational stage of development, the child's ability to visualize the division of a geometrical figure into smaller parts is limited to a finite number of iterations. Only in the stage of formal logical thinking, at around 11-12 years of age, is a child able to envision subdivision as an infinite process. In contrast to this belief, the results of the Taback study, which was also based on a geometrical problem (the questions used were subdivisions) (1975; in Monaghan, 2001), show that children aged 10-12 – with some rare exceptions – still do not understand the concept of infinity. In his study, Fischbein (1979) researched the development of the intuition of infinity. 'The intuition of infinity means what we really feel as being true or self evident concerning the magnitude (the numerosity, the power) of infinite sets, and not what we accept as being true as a consequence of a logical, explicit analysis' (p. 33). He finds that the intuition of infinity appears to be relatively stable from 12 years of age onward, but that the proportion of finitist interpretations is still higher than that of infinitist interpretations (about 60%: 40%). The results of a study by Hannula et al (2006a, 2006b) show that in the fifth grade (when children are 11-12 years old) most students have no concept of infinity, and the situation is not much better in the seventh grade (when they are 13-14 years old). Monaghan (2001, p. 244), who examined pre-university 16-18 years old students' conceptions of limits and infinity, finds that: 'Students' primary focus on infinity was as a process, something which goes on and on. An'object' view of infinity (which does not prescribe a process view in these students) was ascribed to some students, through reference to a very large number or cognizance of collections containing more than any finite number of elements.' As Fischbein (2001) says:'What our intelligence finds difficult, even impossible, to grasp is actual infinity: the infinity of the world, the infinity of the number of points in a segment, the infinity of real numbers…'

Reasons for difficulties in the understanding of infinity

One of the main obstacles in children's understanding of infinity is its abstract nature – the concept of infinity is difficult to link to real-life experiences and is therefore dependant on our ability to visualise mentally. According to Fischbein (1979), the main source of difficulties which accompany the concept of infinity is the fundamental contradiction between this concept and our intellectual schemes, which are naturally adapted to finite realities. Monaghan (2001) points out the fact that the real world is apparently finite and there are thus no real referents for discourse regarding the infinite. The problem in understanding the concept of infinity also stems from the fact that 'the mathematical world is a non-temporal world where infinite summations can be done without reference to time. Outside of the world of pure mathematics the expression such as "going on forever" would be meaningless for a child because no process exists which could last forever' (Monaghan, 2001).

The results of the above studies show that the next difficulty in the process of understanding the concept of infinity presents itself in the stage when children can already imagine that the process of counting never ends, but are not capable of gaining insight into the set of all natural numbers as a set of all numbers one gets when applying the same process into infinity. Moreover, even if we understand the aspect of actual infinity, we might stumble on problems when trying to decipher some paradoxes related to the concept of infinity. One of the most well-known paradoxes is Zeno's paradox from 5th century BC about the footrace between Achilles and the tortoise (see McLaughlin, 1994). The tortoise gets a head start, but even though Achilles moves much faster than the tortoise, he can never pass it. Why? Achilles must first reach the tortoise's starting point. During this time the tortoise moves further on and Achilles has to reach the new starting point where the tortoise has already been. If we apply the same procedure ad infinitum, we realize that Achilles cannot overtake the tortoise since there is an infinite number of such starting points and at each of them the tortoise will be a step ahead of him.

This and other paradoxes have been a thorn in the side of many a mathematician throughout history. Especially challenging were problems dealing with the comparison of infinite sets. Even Galileo (1631; in Jahnke, 2001) pointed out contradictions in comparing the set of natural numbers and squares of natural numbers: on the one hand, it is possible to assign to every natural number exactly one square and vice-versa, which means that the set of natural numbers and the set of their squares are equivalent; however, on the other hand, the set of the squares is a part of the set of natural numbers.

Cantor (1845-1918) was the mathematician who finally put into practice a new understanding of an infinite set. His theory of sets upset the established way of thinking about finite sets, a manner of thinking that failed when applied to infinite sets. An infinite subset of an infinite set does not necessarily have a lower cardinality. To compare the sizes of infinite sets, Cantor established the criterion of determining a 'pairwise correspondence' (Jahnke, 2001). He also proved that more than one kind of infinity exists: instead of one concept of infinity corresponding to our intuitive understanding of endlessness, an infinite possible world of infinities exists (Fischbein, 1979).

Cantor's theory introduced a very formalistic manner of understanding sets and infinity, and because it contradicts natural logic this manner is foreign to the common person.

Factors influencing the perception of the concept of infinity

In the continuation, we shall try to shed some light on the concept of infinity from different aspects.The results of many studies (Hannula et al, 2006; Tsamir, 2001; Monaghan, 2001) show that the perception of infinity depends on the context (based on either numbers or geometry), on the type of the infinite set (represented by the aspects of infinitely large, infinitely many or infinitely close) and also on the representation of the problem.

Let us take the above mentioned problem of comparing the set of natural numbers and the set of their squares as a starting point. Tsamir (2001) and Duval (1983) established that the method of representation can substantially contribute to children's insight into the fact that two sets of seemingly different sizes are in fact of the same cardinality.

Let us compare set A which contains all natural numbers with the set containing the squares of all natural numbers (Tsamir, 2001). Tsamir presented her students with four interpretations of the two sets:

a) horizontal representation

A = {1, 2, 3, 4, 5, …} B = { 1, 4, 9, 16, 25, …}

b) vertical representation

A = {1, 2, 3, 4, 5, …}

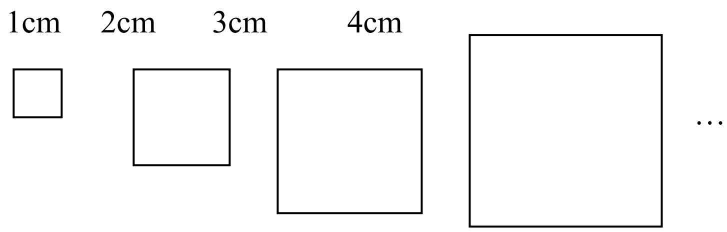
B = {1, 4, 9, 16, 25, …}

c) numeric – explicit representation

A = {1, 2, 3, 4, 5, …}

B = {1^2, 2^2, 3^2, 4^2, 5^2 …}

d) geometric representation



where set A is the set of all sides of squares from the above sequence, and set B is the set of all areas of the above squares.

The results show that the numeric-explicit and the geometric representations encourage one to one correspondence substantially more than the horizontal representation does. We can summarize that the understanding of infinity can be improved through the use of suitable representations. Suitable representations are those that trigger a cognitive conflict related to a student's existing, intuitive perception of infinite sets – that is, a conflict that is mostly based on the concept of potential infinity. ‘Even students have no special tendency to use the Cantorian method of “one to one correspondence”, they are prone to visual cues that highlight the correspondence’ (Hannula et al, 2006a, p. 4).

Empirical study

Problem definition and methodology

In the empirical part of the article, we present a part of an extensive study conducted with primary teacher students with the aim of researching their understanding of the concept of infinity. We were mostly interested in finding out how students who received no in-depth instruction on abstract mathematical content such as Cantor’s set theory tackle problems related to the comparison of infinite sets and what argumentation they provide for their answers.

The empirical study was based on the descriptive, non-experimental method of pedagogical research.

Research questions

1. How successful are students when solving problems that require them to use the concept of infinity?
2. How do students compare different infinite sets (one set as a proper subset of the other vs. pairwise correspondence)?
3. Do different representations of a problem on infinity help students to establish the relation between infinite sets?
4. Do students know how to use their knowledge about the relationships between infinite sets in a new situation?
5. What type of arguments and language do students use when substantiating and explaining problems related to infinity?

Sample description

The study was conducted in May 2009 at the Faculty of Education, University of Ljubljana, Ljubljana, Slovenia. It encompassed 93 third-year students of the Primary School Teaching Programme.

Data processing procedure

The aims of the study were examined on the basis of mathematical problems about infinity which were to be solved, and their solutions supported with arguments. The data gathered from mathematical tests were statistically processed by employing descriptive and inferential statistical methods (frequency and chi-square distribution). Students' argumentations of their answers to mathematical problems were also qualitatively processed and their answers classified into meaningful categories.

Results and interpretation

The sequence of problems that demanded a comparison of two infinite sets, one of which includes the other, was meant to investigate whether the students know that all pairs of sets have the same cardinality.

Problems with the same content differed only in the method of representation (symbols - vertically, graphical representation, symbols - horizontally ...).
One part of the instructions was identical for all problems.
Compare the cardinality of the following sets of numbers. For each pair of sets, circle the set you believe has a larger cardinality. If you think that the sets have the same cardinality, circle both sets. Provide arguments for your choice.

As already mentioned, our respondents received no in-depth theoretical mathematical instruction required for tackling the concept of infinity. The term 'cardinality' was known to them only in relation to comparing finite sets. The current Slovene educational system provides students with two different methods that can be used to compare finite sets: the method of counting and the method of one to one correspondence. The latter is taught to children in the first grade of primary school before the introduction of numbers. After that it is no longer present in the curriculum until they start attending university and learning about comparisons of infinite sets. In this case the method of counting obviously fails, which is why we set to explore whether our students were able to recognise the advantages of using one to one correspondence as a way of comparing two infinite sets.

Let us see the results and the arguments students gave for individual problems.

Problem 1 *(written with symbols, vertically)*:

a) 1 2 3 4 5 ...
b) 2 4 6 8 10 ...

The sets above were illustrated by means of symbols, one above the other, so that the respondents could consider the possibility of a one to one correspondence and come to the conclusion that both have the same number of elements.

Answers	Number of answers	Share of answers
Both sets have the same cardinality.	22	24%
The set of natural numbers has a larger cardinality than the set of even numbers.	63	68%
The set of even numbers has a larger cardinality than the set of natural numbers.	1	1%
No answer	7	7%

Table 1: Results to Problem 1

The arguments for the set of natural numbers having a larger cardinality than the set of even numbers were the following:

- because even numbers are a subset of natural numbers (13)
- because an even number is every other number, while natural numbers come one after another (none of them is left out) (25)
- both sets are infinite, but there are twice as many natural numbers (3)
- because natural numbers follow each other to infinity (1)
- because the first number is 1 (1)
- 20 respondents provided no explanation.

Only one respondent decided that there are more even numbers by providing the explanation that even numbers can also be negative.

The respondents who answered that both sets have the same cardinality provided the following arguments:

- both are infinite (21)
- every odd number has its even pair.

There were also two explanations that both sets are infinite, but nothing can be said as to which of the two has a larger cardinality.

It becomes evident that the written representation of the two sequences (one below the other) did not encourage the students to consider the possibility that the one to one correspondence of elements indicates that both sets are infinite and of the same cardinality.

Problem 2

The problem included two different examples.

Problem 2.1 *(written with symbols, vertically explicit):*

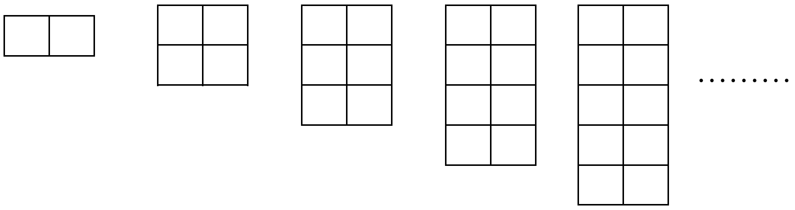
a) 1 2 3 4 5…

b) $2 \cdot 1$ $2 \cdot 2$ $2 \cdot 3$ $2 \cdot 4$ $2 \cdot 5$ …

Problem 2.2 *(graphical representation of sets)*

a) set A = {height in the sequence of lower rectangles}

b) set B = {areas of respective rectangles}



- 38 respondents compared the cardinalities of sets of natural and even numbers where even numbers were presented as products 2 x n (where n is a natural number). The two sets were written in rows, one below the other, n in the first line above n in the second line.
- 55 respondents compared the cardinalities of sets of natural and even numbers where the set of natural numbers was presented as the heights of the rectangles and the set of even numbers as the set of their respective areas.

What both problems have in common is that the presentation of the sets explicitly implies the existence of a bijective correspondence between the elements of the sets. We wanted to determine whether such explicit presentations could contribute to students' recognition of the two sets having equal cardinality.

Problem 2.1

Answers	Number of answers	Share of answers
Both sets have the same cardinality.	6	16%
The set of natural numbers has a larger cardinality than the set of even numbers.	22	58%
The set of even numbers has a larger cardinality than the set of natural numbers.	7	18%
No answer	3	8%

Table 2: Results to Problem 2.1

The arguments for both sets having the same cardinality were the following:

- both are infinite (4)
- no explanation provided (2).

Those who decided that the set of natural numbers has a larger cardinality supported their claims with the following arguments:

- because even numbers are a subset of natural numbers (2)

- because the set of natural numbers has more numbers, they come one after another and only every second number is an even number (10)
- because it is not clear where even numbers end (1)
- because with even numbers, numbers grow very fast (1)
- because the set of even numbers does not include the number 1 (1)
- no explanation provided (7).

The respondents who claimed that the set of even numbers has a larger cardinality than the set of natural numbers provided the following reasons:

- because it is more precise (1)
- no explanation provided (6).

<u>Problem 2.2</u>

Answers	Number of answers	Share of answers
Both sets have the same cardinality	12	22%
The set of natural numbers has a larger cardinality than the set of even numbers.	3	5%
The set of even numbers has a larger cardinality than the set of natural numbers.	14	26%
No answer	26 (2)	47%

Table 3: Results to Problem 2.2

The arguments for both sets having the same cardinality were the following:

- they are of the same size because if we do not increase the height, the area does not increase either (1)
- the area increases with height (3)
- the same; there are as many rectangles in set b as there are heights in set a (1)
- because the areas increase proportionally to heights (1)
- both are infinite (8)
- no explanation provided (1).

Those who decided that the set of areas has a larger cardinality supported their claim with the following arguments:

- the area represents more than height (13)
- no explanation provided (1)

The following arguments were provided for the claim that the set of heights has a larger cardinality than the set of areas:

- even numbers are a subset of natural numbers (1)
- two respondents provided no explanation.

Problem 3 *(written with symbols, horizontally)*

a) 2 4 6 8 10 … b) 1 2 3 4 5 …

The respondents compared the sets of natural and even numbers, represented by symbols written horizontally and no longer vertically as in Problem 1. We assumed that if the respondents notice the correspondence of elements of one set with the elements of the other, they would no longer need them to be written down vertically in order to provide a correct answer to the problem.

Answers	Number of answers	Share of answers
Both sets have the same cardinality	22	24%
The set of natural numbers has a larger cardinality than the set of even numbers.	61	66%
The set of even numbers has a larger cardinality than the set of natural numbers.	0	0%
No answer	10	10%

Table 3: Results to Problem 3

21 of 22 respondents who claimed that both sets have the same cardinality provided only the explanation that both are infinite, of the same size. One respondent listed no arguments.

The arguments for the set of natural numbers having a larger cardinality than the set of even numbers were the following:

- every other number is an even number (16)
- even numbers are a subset of natural numbers (9)
- these numbers follow each other in a sequence (4)
- there are more natural numbers than even numbers (2)
- because they start with 1 (1)
- both sets are infinite, but there are twice as many natural numbers (1)
- no explanation provided (29).

Among the respondents who circled none of the sets, 3 familiar argumentations can be found: both sets are infinite, but nothing can be said as to which of them is larger.

Problem 4 *(transfer of knowledge to a new situation):*

a) infinite number of millimetres b) infinite number of kilometres

This problem asked respondents to compare the following sets: an infinite number of millimetres and an infinite number of kilometres. We can observe that the problem is similar to the previous problems since in all cases two infinite sets are compared and one of the sets is contained in the other (the set of even numbers being a subset of natural numbers and millimetres being the smaller unit that can also be converted to kilometres). This problem is different from the previous ones in that it uses the expression 'infinite' in front of the words millimetre and kilometre.

Answers	Number of answers	Share of answers
Both sets are of equal size.	55	59%
An infinite number of millimetres is more than an infinite number of kilometres.	17	18%
An infinite number of kilometres is more than an infinite number of millimetres.	6	7%
No answer	15	16%

Table 4: Results to Problem 4

We can observe that 55 respondents decided that the sets are infinite and that they have the same cardinality. Their explanations were straightforward: both sets are infinite (43), infinite equals infinite (6). The word 'infinite' which was used in the problem had a key role in their argumentation. 6 respondents provided no explanation.

The argumentations for an infinite number of millimetres being more than an infinite number of kilometres were as follows:

- since mm is a smaller unit than km, there are more millimetres (8)
- no explanation provided (9).

The respondents who claimed that an infinite number of kilometres is more than an infinite number of millimetres, referred to the fact that 1 km is more than 1 mm (2 respondents), while 4 respondents provided no explanation for their answer.

Three respondents did not circle any of the sets, but they still provided the following explanation: though both sets are infinite, nothing can be said about which of them is larger.

The table below is a summary of conclusions for the individual problems. Each of the five columns contains information on the share of correct answers as well as shares of students who believe that the set of natural numbers has a larger cardinality than the set of even numbers.

	Even/natural (vertical): Problem 1	Even/natural (vertical explicit): Problem 2.1	Even/natural (graphical): Problem 2. 2	Even/natural (horizontal): Problem 3	Infinite mm/ infinite km: Problem 4
Share of correct answers	24%	16%	22%	24%	59%
Share of those who chose natural numbers	68%	58%	5%	66%	
Share of those who chose even numbers	1%	18%	26%	0%	
No answer	7%	8%	47%	10%	

Table 5: Comparison of answers to different problems

As the table shows, the share of respondents who provided correct answers to the problem of comparing the sets of even and natural numbers is small and more or less constant. No conclusions can be drawn regarding the influence of different representations on individual respondents' answers, i.e. whether the similar shares for different representations of the same problem also mean that the students did not change their answers depending on the representation. The answer can be found in the table below, which contains the data on the number of respondents who changed their answers. The table lists the number of students whose answers changed from incorrect to correct depending on the representation of the problem. The χ^2-test was used to confirm whether the transitions from incorrect to correct are statistically relevant.

		Number of changed answers/number of respondents	Share of changed answers	Number of changes from an incorrect to a correct answer	P
Problem 1	Problem 2.1	6/38	16%	0	1.00
Problem 1	Problem 2.2	41/55	75%	7	0.34
Problem 1	Problem 3	10/93	11%	4	1.00

Table 6: The influence of representation on the number of correct answers

We should note that the geometrical problem 2.2 had the strongest influence on the changes in answers since 75% of respondents answered it differently as the previous problem. Unfortunately the majority of the different answers to problem 2.2 are due to students who provided no answer (see Table 5). There are 7 changes in answers from incorrect to correct, which is not statistically relevant (see Table 6). The data in the last column of Table 6 can lead us to the conclusion that none of the representations in 2.1, 2.2 and 3 had a statistically significant influence on the correct answers to the problem of comparing the cardinality of the sets of even and natural numbers.

Comments

Based on the results of the statistical analysis, the following can be concluded:

- The share of students who believe that the comparison of the sets of natural and even numbers involves two sets of the same cardinality is low. The majority believe that the set of natural numbers has a larger cardinality than its proper subset.
- It does not appear that the method of representation has any influence on the perception of the relationship between the two sets (Problem 1: Problem 3 (P=1.00)).
- Comparison of geometric and vertical-explicit representations: it seems that they do not have a significant impact on students' thinking about the relations between the sets (even and natural), since even after the introduction of these two methods of representation, the results do not improve (cf. problem 1 and 3).
- However, the majority of respondents have difficulties recognizing a geometrically represented relationship between the cardinality of even and natural numbers as a variation of the problem of comparing the sets of even and natural numbers. They did not notice that even and natural numbers are represented by the heights and areas of rectangles respectively.
- Comparing even/natural (Problems 1, 2 and 3) and infinite mm/infinite km (Problem 4): in both cases infinite sets are compared, where we have a kind of inclusion of one set into another – it seems that the word infinite encouraged students to decide that the sets are equal.

Based on the argumentations that the respondents provided for each of their answers, we can conclude:

- Students who answered correctly mostly claimed that both sets are infinite.
  We see this argument as insufficient: the fact that two sets are infinite does not enable the conclusion that the sets also have the same cardinality. Cantor proved that more than one kind of infinity exists. Even more, the sets with cardinality aleph0, which are equipollent to the set of natural numbers, represent only one kind of infinite sets in the infinite world of infinities.
- None of them based their answer on the existence of the bijective correspondence between the two sets.

- The idea of one to one correspondence was indicated only by some explanations to geometrical problem:

  'They are both equal because if we do not increase the heights, the corresponding areas do not increase either.' (1)

  'They are both (heights and areas) increasing together.' (4)

  'Equal: the number of rectangles is equal to the number of heights.' (1)
- Students' explanations of wrong answers are usually based on the fact that there are more natural numbers than even ones (the most common answer: 'even numbers are a subset of natural ones').

Summary

The concept of infinity is without a doubt one of the most abstract notions that primary school pupils encounter in mathematics classes. The reasons for this are difficult to define. They can be described as a set of factors such as the lack of tangible models representing the concept in everyday life, the contradiction between the concept and its use in everyday discussions, and also, within the field of mathematics, the disparity between the approach to this notion and the mathematical approach to finite entities. The article focuses on a problem-based situation in which two infinite sets are compared, one of them being a subset of the other. It would be logical to assume that the latter has a lower cardinality, but this is not the case. In order to understand this, we need a certain theoretical mathematical knowledge. We aimed to establish whether we already possess some rudiments of this knowledge and whether they resemble the type of deduction used in everyday mathematical and non-mathematical situations. We also researched whether the method of representation can bring about a better understanding of the problem.

Our findings show that different methods of representation of the problem did not lead the respondents to the insight that the compared sets of even and natural numbers have the same cardinality. Some of the respondents intuitively felt this, but they seemed to lack the mathematical knowledge required to provide proper argumentations for their assumptions. Their explanations were poor and mostly founded on the fact that both sets are infinite. None of them chose to use the method of one to one correspondence (creating pairs of elements), which is the essence of Cantor's theory.

The results suggest that Tsamir's (2001) findings that an appropriate selection of representation methods can improve students' understanding of the problem cannot be confirmed. Still, there were some important differences between the two studies. In P. Tsamir's study, teachers

actively participated in the process of learning. They attempted to raise students' awareness that they were being presented with the same problem, albeit in different representations. Our method of data collection did not include active work with the students; it merely assessed their understanding of the problem. The results presented are a product of the students' own work, without any outside factors guiding them or triggering a cognitive conflict by means of existing answers. We can therefore conclude that the respondents possess some rudimentary ability to understand abstract problems on infinity. However, without some external help, i.e. someone guiding their reasoning and warning them about the contradictions between the answers to the problems, they are not able to arrive at the right conclusions without a great deal of effort. The finding is not surprising if we consider the fact that mathematicians, philosophers and other thinkers have wrestled with the problems related to the notion of infinity since Aristotle's time, and not until the 19th century was Cantor able to form a consistent theory that overcame the reservations of the mathematical community.

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Misleading strategies used in a non-standard division problem

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Abstract

Here we focus on Finnish pre-service elementary teachers' (N = 269) and upper secondary students' (N = 1434) understanding of division. In the questionnaire, we used the following non-standard division problem: "We know that 498 : 6 = 83. How could you conclude from this relationship (without using long division algorithm) what is 491: 6 = ?" This problem measures especially conceptual understanding, and partially also adaptive reasoning and procedural fluency. Based on the results we can conclude that division seems not to be fully understood: Only 45 % of the pre-service teachers and 37 % of upper secondary students were able to produce complete or mainly correct solution. In this paper we focus on errors based on lack of conceptual understanding.

Introduction

In this paper we concentrate on pre-service teachers' and upper secondary students' understanding of division and especially on the mistakes they have done. The paper form a part of a larger research project "Elementary teachers' mathematics" financed by the Academy of Finland (project #8201695), in which data were collected at three Finnish universities (Helsinki, Turku, Lapland). As part of the project we also collected comparison data from 1434 upper secondary students (grade 11, average age 17-18 years) from 34 Finnish schools selected at random.

In Finland elementary teacher education program is very popular and highly regarded: only some 10-15 % of the applicants can be admitted to the program. Students in Finnish elementary teacher education take a Master's Degree in education. Yet, not all pre-service elementary teachers have the level of proficiency in mathematics at the beginning of their studies that will best serve their future career needs (e.g. Merenluoto & Pehkonen 2002).

In the curriculum for Finnish comprehensive school (NBE 2004) one of the principal goals already in the second grade is that pupils should master and understand basic calculations. But earlier studies show that also pre-service teachers and upper secondary students have clear weaknesses in understanding of division, not only in Finland (Merenluoto & Pehkonen 2002), but also in other countries (e.g. Simon 1993, Campbell 1996). One of the main reasons for these weaknesses seems to be that pre-service teachers have primitive models of division (e.g. Graeber & al. 1989; Simon 1993). Even after learners at school have had formal, algorithmic teaching, they continue to be influenced by primitive partitive and quotitive models (Fischbein & al. 1985).

Pre-service teachers' and upper secondary students' understanding on division has usually been measured with tasks involving real-world contexts (see e.g. Graeber & al. 1989; Silver & al. 1993, Simon 1993) or an abstract context, and in both contexts students have been allowed to use a calculator or long division as an aid (see e.g. Simon 1993; Campbell 1996; Zazkis & Campbell 1996). The non-standard division task with an abstract context we use in this study differs specifically from the tasks used in earlier studies in that 1) participants must use a given equation as a starting point for their reasoning and 2) may not use the long division algorithm or a calculator when solving the task.

THEORETICAL FRAMEWORK

Mathematical understanding and mathematical proficiency

Mathematical understanding can be characterized as a continuous process that is fixed to a certain person, a mathematical content domain and a special environment (cf. Hiebert & Carpenter 1992). Mathematical understanding answers the question "Why?" and, in addition, entails, among other factors, the skills required to analyze mathematical statements. Within the last twenty years, researchers have developed theories on mathematical understanding as a dynamic process, i.e. how an individual's mathematical understanding develops (e.g. Pirie & Kieren 1994). In the Pirie & Kieren model, understanding is seen as a process where the individual can progress from one level of understanding to the next one. But the progress from level to level is not necessarily linear; an individual may also regress in his or her understanding.

In the research literature, mathematical proficiency is often defined as procedural knowledge and conceptual understanding (cf. Hiebert & Lefevre 1986). In this paper we will use the following, more detailed classification adopted by Kilpatrick (2001, 106):

> "The five strands of mathematical proficiency are (a) *conceptual understanding,* which refers to the student's comprehension of mathematical concepts, operations, and relations; (b) *procedural fluency,* or the student's skill in carrying out mathematical procedures flexibly, accurately, efficiently, and appropriately; (c) *strategic competence,* the student's ability to formulate, represent, and solve mathematical problems; (d) *adaptive reasoning,* the capacity for logical thought and for reflection on, explanation of, and justification of mathematical arguments; and (e) *productive disposition,*

which includes the student's habitual inclination to see mathematics as a sensible, useful, and worthwhile subject to be learned, coupled with a belief in the value of diligent work and in one's own efficacy as a doer of mathematics."

The division task used in this study measures several of the strands of mathematical proficiency mentioned by Kilpatrick (2001), but especially it measures conceptual understanding, partially also adaptive reasoning and procedural fluency.

Understanding division

Division is an important but complex arithmetical operation to consider in elementary teacher education. There are many reasons for its complexity: 1) division is taught as the inverse of multiplication, so understanding of division requires good understanding of multiplication; 2) division involving big numbers requires good estimation skills; 3) within the models of equal groups and equal measures two aspects of division can be differentiated: quotitive division (how many sevens there are in 21) and partitive division (21 divided by 7). (e.g. Anghileri & al. 2002)

People use very different strategies in solving division problems. Some of them are useful and some are misleading. Prior research has identified the following useful strategies (e.g. Heirdsfield & al. 1999): 1) Several different counting strategies: skip counting, repeated addition and subtraction, chunks; 2) Using a basic fact; 3) Holistic strategies.

In a study by Graeber & al. (1989), 129 female pre-service teachers had high scores on solving verbal problems involving the partitive model of division. They were less successful on the quotitive division problems, and these primitive models influence pre-service teachers' choice of operations. Primitive models seem to reflect an understanding whereby a student separates things into equal size groups. In Simon's (1993) study of pre-service elementary teachers the whole-number part of the quotient, the fractional part of the quotient, the remainder, and the products generated in long division did not seem to be connected with a concrete notion of what it means to divide a quantity.

Campbell (1996) studied 21 pre-service elementary teachers' understanding of division with remainder. He conducted clinical interviews with the students, who tried to solve four tasks with abstract contexts. The task we use here has some similarities in contrast to the following task used by Campbell (1996, 179): "Consider the number $6 \cdot 147 + 1$, which we will refer to as A. If you divide A by 6, what is the remainder? What is the quotient?" In Campbell's (1996, 182-183) study of the 19 participants who tried to solve this task, 15 calculated the dividend although it entailed additional trouble. Of those 15 respondents 9 calculated the dividend and relied upon long division in solving the task. Of those 4 who did not calculate the dividend, only 2 correctly identified the remainder and the quotient.

Zazkis & Campbell (1996) investigated 21 pre-service elementary school teachers' understanding of divisibility and the multiplicative structure of natural numbers in an abstract context. The following is an example of the tasks used: "Consider the numbers 12 358 and 12 368. Is there a number between these two numbers that is divisible by 7 or by 12?" Many pre-service teachers used long division as the procedural activity, but some degree of conceptual understanding was evident as well.

In a study by Silver & al. (1993), a total of 195 sixth, seventh and eighth graders from a large middle school solved three quotient division problems involving remainders with a real-world context (the number of the buses needed). The symbol forms of the word problems were a) 540:40; b) 532:40 and c) 554:40. Of the respondents, 91 % used appropriate procedures, and 73 % of them applied long division. Only 43 % of the participants understood that the result - the number of buses – was an integer.

Focus of the paper

In this paper we focus on the following research question: How the lack of conceptual understanding is reflected in pre-service elementary teachers' and upper secondary students' strategies when solving a certain non-standard division task?

EMPIRICAL RESEARCH

Research participants and data

The study forms a part of the research project "Elementary teachers' mathematics" financed by the Academy of Finland (project #8201695) . in which data were collected on 269 pre-service elementary teachers at three Finnish universities (Helsinki, Turku, Lapland). Two questionnaires were administered in autumn 2003 to assess the pre-service teachers' knowledge, attitudes and skills in mathematics at the beginning of their mathematics education course. Students had 60 minutes time for the questionnaires and were not allowed to use calculators. The aim of the questionnaires was to measure their experiences of mathematics, their views of mathematics and their mathematical proficiency. As part of the project we also collected comparison data from 1434 upper secondary students (grade 11, average age 17-18 years) from 34 Finnish schools selected at random. The project is described in detail e.g. in the published paper of Kaasila & al. (2008).

The initial proficiency test contained a total of 12 mathematical tasks. The focal content areas were the rational numbers and related operations (in particular division), because previous research indicates that these are problem areas (e.g. Hannula & al. 2002). All in all, the initial proficiency test focused on content knowledge different from that tested usually in upper secondary school and on the mathematics component of the matriculation examination.

In conjunction with the project we also collected comparison data with the same questionnaires from upper secondary school. Altogether 50 schools were selected at random from all Finnish upper secondary schools. A letter was sent to the directors of the schools in the sample, in which they were asked to select from their school one group of students in the general course and one in the advanced course in second-year mathematics. We received responses from 34 schools representing a total of 65 student groups. Thus, we obtained in total data from 1434 students.

The non-standard division task we used is the following:

> "We know that 498 : 6 = 83. How could you conclude from this relationship (without using the long division algorithm), what is 491: 6 = ?"

Analysis

We did not find in the research literature a task similar to the one used in this study. As mentioned earlier, our task shares certain features with that used by Campbell (1996). However, it also differs in a number of respects: Firstly, in the task used by Campbell, the dividend is explicitly mentioned as the 'right hand side' of the division algorithm, whereby respondents have an opportunity to directly identify the quotient and the remainder. In our task, the starting equation is given in the form of division and does not involve a remainder. Secondly, unlike Campbell, we do not mention in the context of our task the concepts of remainder and quotient. Thirdly, the participants in our study did not have permission to use the long division algorithm or a calculator, which were central aids in Campbell's study.

In the first phase of this study (see Kaasila & al. 2005) we broke the 269 pre-service elementary teachers' solutions down into main categories and subcategories by applying analytic induction (cf. LeCompte, Preissle & Tesch 1993). This phase was very data-driven. After reading carefully part of the participants' solutions, we constructed a framework for categories.

Then we tested if the other strategies including our data fitted with this framework. If some solution did not match our category framework, we modified the framework on the basis of this solution.

In the second phase of the study (cf. Hellinen & Pehkonen 2008), a deductive approach was used: The 1434 upper secondary students' solutions were categorized using essentially the same classification as used in the first phase when analyzing pre-service elementary teachers' solutions. A number of categories were identified in addition to those formed in the first phase.

In the third phase we harmonized the categories we found in the phases one and two by reanalyzing a part of the pre-service elementary teachers' solutions. Finally we compared the pre-service elementary teachers' reasoning (or solution) strategies with the upper secondary students' reasoning strategies.

RESULTS

When presenting our results, we classified the participants' strategies mainly on the basis of their degree of conceptual understanding. About 30 % of the pre-service teachers and of the upper secondary students produced either rigorous and complete solutions or correct solutions with missing elements in justification. In addition, 15 % of the pre-service teachers and 7 % of the upper secondary students seemed to understand the task but made a careless mistake. So in all, 45 % of the pre-service teachers and 37 % of the upper secondary students were able to produce complete or mainly correct solution. The rest of the participants made errors based on the lack of understanding. More details on all results can be found in the published paper of Kaasila & al. (2009).

In this paper we focus on participants' errors based on lack of understanding.

Table 1. Main categories of erroneous strategies based on lack of understanding used by the pre-service teachers (PST, N = 269) and the upper secondary students (USS, N = 1434).

	PST	%	USS	%
Errors based on lack of conceptual understanding	146	55	907	63
1 Thinking limited to integers	59	22	167	12
2 Clear misconception	12	5	44	3
3 Other mistakes / irrelevant strategies	75	28	696	48

Of pre-service teachers 55 % and of upper secondary students 63 % produced an erroneous solution that showed at least some lack of conceptual understanding. We divided these strategies into three subcategories, in some cases these are still divided into smaller groups.

1 Thinking limited to integers: 22 % of the pre-service teachers and 12 % of the upper secondary students were not able to calculate [determine] the quotient. We divided these strategies into two subcategories:

1.1 The respondent knew that the answer was not an integer, but he/she was not able to deal with the remainder (12 % vs. 9 %):

Example: The number 491 is 7 units smaller than 498. Therefore 6 should go one time fewer into 491. I can't think of any explanation for the fact that 6 goes into 491 only 81 times. (3016)

1.2 The answer was given as an integer (10 % vs. 3 %). In these cases the respondents are not sure if the answer might be something other than an integer.

Example: The number 491 is 7 units smaller than 498. Consequently, the answer is 82, but one unit is left over... But perhaps it can be ignored or should the answer be a decimal? (3055)

2 A clear misconception: 5 % of the pre-service teachers and 3 % of the upper secondary students had clear misconceptions in their answers. We divide these into two subcategories:

2.1 The remainder was considered as a decimal (tenths) instead of sixths (3 % vs. 1 %).

Example: 498 - 491 = 7; 7 - 6 = 1. The result is 82.1 (1012)

2.2 The respondent subtracted the difference of the dividends from the quotient (2 % vs. 2 %): In these cases, the respondents seemed to understand division such that the dividend and the quotient change at the same rate. The use of this strategy indicates major problems of mathematical understanding.

Example: 498 – 491 = 7; 83 – 7 = 76. (3057)

3 Other mistakes / irrelevant strategies: 28 % of the pre-service teachers and 55 % of the upper secondary students obtained no answer at all or presented a solution that was not relevant to the research. These cases are grouped into three subcategories:

3.1 The answer was reached by experimenting or in some way without using the connection given in the task (1 % vs. 2 %): This type of reasoning strategy usually produced erroneous results, but there were also correct ones.

Example: 6 goes 50 times to 300, this leaves 191 => where 30 · 6 = 180; this then leaves 11, which into which 6 goes almost 2 times; thus 491 : 6 => 50 + 30 + 2 (almost) ≈ 82 times (5687)

3.2 Irrelevant reasoning, an inaccurate response (10 % vs. 23 %): In this category, the results were very inaccurate and reasoning irrelevant.

3.3 No result, a kind of attempt (17 % vs. 30 %): The respondents in this category did not produce any result, or anything that made sense.

Example: I can't do it without a calculator (3079).

DISCUSSION

The non-standard division task used in this study was quite a challenging one because it measured many central strands of mathematical proficiency mentioned by Kilpatrick (2001), especially conceptual understanding, but partly also adaptive reasoning and procedural fluency. More than half of the participants either produced no result at all or used erroneous strategies. In this study we classified the participants' solutions mainly on the basis of their conceptual understanding. We are aware that there are also many alternative ways to classify the participants' strategies. The second possibility, for example, would be to emphasize more the participants' adaptive reasoning skills.

We identified four main reasons for erroneous or incomplete solutions: 1) staying on the integer level (difficulties especially in conceptual understanding); 2) inability to handle the remainder of the division (difficulties especially in procedural fluency), 3) difficulties in understanding the relationships between different operations (problems especially in conceptual understanding), and 4) inadequate reasoning strategies (difficulties especially in adaptive reasoning). In the following we will consider these reasons in more detail.

1) Staying on the integer level: Most of the pre-service teachers and the upper secondary students understood that the result was going to be roughly 82. It seems that they could solve the problem 492:6 =? by using the relation (basic fact) between division and subtraction, or division and multiplication, and/or derived facts (cf. Heirdsfield & al. 1999; Neuman 1999). The difficulty of the problem "491:6 =?" can be summed up in the following sentence: "What should one do with the one extra unit?" 10 % of the pre-service teachers and 3 % of the upper secondary student gave their answer as an integer, and it seems that in these cases they did not

even think that the answer might be something other than an integer. Some respondents mechanically subtracted 7 (or 6) directly from 83. This kind of thinking suggests a major shortcoming in their understanding of division.

2) Inability to handle the remainder: Some of the respondents seemed to understand that the result was not an integer but a fraction (or a decimal fraction), but they could not handle the remainder. For example, they expressed the remainder in the answer in tenths not in sixths. These respondents seem to master or prefer more tenths than other fractions (cf. Campbell 1996, 180). According to the Finnish curriculum (NBE 2004), concept of a number is enlarged to include fractions already in the lower grades of the elementary school. Moreover, in Finland elementary teacher students have been selected from a pool of applicants that is about ten times larger than the group ultimately admitted, although mathematics is not one of the admissions criteria (cf. Kaasila & al. 2008).

It seems that a majority of the participants experienced difficulties in assimilating and accommodating the meanings of quotient and remainder in a less familiar “situational” context of the task. Partitive dispositions towards division exacerbate many difficulties that quotitive dispositions towards whole number division with remainder may resolve. (Campbell 1996) It seems that in school dealing with remainders has been a procedural matter, with too little attention focused on the idea that the fractional part of the quotient provides different (yet related) information from the remainder (Simon 1993).

3) Difficulties in conceptual understanding of the relationships between different operations: In order to solve the non-standard division task, respondents needed to understand the relationship between division and subtraction or the relationship between division and multiplication. Their minimal use of multiplication is a rather surprising result. According to Anghileri & al. (1999), working simultaneously with multiplication and division help pupils to understand both operations and their connection better, i.e. that division and multiplication are inverse operations of one another. Our results suggest that most elementary teacher students rely on standard algorithms. Although some of the students understood divisibility, they did not necessarily understand the nature of whole number division with a remainder and its relation to multiplication (cf. Zazkis & Campbell 1996).

4) Insufficient reasoning strategies: Of pre-service teachers 10 % and of upper secondary students 14 % produced a correct solution with missing elements in justification. The reason for insufficient reasoning strategies may be a lack of language skills, because the respondents had

great difficulties in providing written explanations of their reasoning (cf. also Silver & al. 1993). Although the participants in question could solve the task and obtain the correct result, they were not able to express the operations needed. The Finnish Matriculation Examination Board has also expressed its concern over the lack of reasoning skills on the examination (cf. Lahtinen 2006).

On the basis of this study we can suggest some guidelines for the content of mathematics courses in teacher education and in comprehensive and upper secondary school: Learners need a) a concrete, contextualised knowledge of division and b) conceptual understanding to examine division as an abstract mathematical object (cf. Simon 1993), and especially to understand the relationships between division and the other operations. Above all, learners need in all school grades c) tasks and situations through which they can develop their adaptive reasoning skills. According to our study, a lack of reasoning skills may be one of the main factors causing students difficulties when solving non-standard division tasks.

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Problem Solving in Business Mathematics

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Abstract

Students who study mathematics as a minor at institutions of higher education should be enabled, too, to become future experts to model and solve non-mathematical problems by means of mathematics. However, mathematics is rather often abused as a tool for selection in order to reduce congestion at universities. Hence, the opportunity is barely used to develop problem solving competence – one of the most important competences. In this paper a problem from economy will be presented and analyzed as an example which can be modeled and treated in many ways by means of mathematics and therefore, it can be, among others, the starting point of forming mathematical concepts.

1. Background: Role of Mathematics as a minor at institutions of higher education

One of the main aims at colleges of higher education is to train persons, who are not only qualified in their special field but who are even up to the job market's expectations in different ways and who are also able to update their knowledge as needed. Therefore, the main focus at colleges of higher education is not only on the development of key competences (cf. Kommission der Europäischen Gemeinschaften, 2005) resp. on the transmission of technical knowledge and professional skills but also on the development of so-called job competences (cf. Wissenschaftsrat, 1999). In this complex situation mathematics as a minor plays a very important part, because it is both a way of thinking and a medium for solving problems. Hence, mathematics can support on the one hand indirectly the developing of key resp. job competences and on the other hand directly the developing of professional skills through modeling and solving technical problems. Curdes (2008) formulated this like follows: "For many applications in technical and economic subjects mathematical knowledge and mathematical skills are needed. In order to apply mathematical knowledge to other areas, whenever you are learning mathematics, you should train your problem solving competence and get a view on mathematics which is oriented to understanding."[1] Having said this in mathematics education at colleges of higher education the main focus should be on problem solving.

On the contrary the reality seems to look differently: It is often used the so-called social function of Mathematics, it is abused as a tool for selection in order to reduce congestion at insti-

[1] Translated by the author of this paper. See the origin text in German by Curdes, 2008, p. 17.

tutions of higher education. Therefore, Mathematics becomes a „horror subject” for teachers and sensible students (cf. Roos, 1998).

Furthermore, it will be demonstrated by a problem from economy how the author can imagine herself a practice-oriented and still mathematical demanding training in mathematics at colleges of higher education.

2. A problem from economy: How to order quantities in an economic optimal way?

The problem presented was taken from a textbook[2] for students at secondary schools with main focus on economy and management:

„An electronic company needs 48000 pieces of 1.5-V-batteries per period (e.g. business-year). A 1.5-V-battery costs 1,-€. The company has to pay additionally 100,-€ per order and 15% of the average inventory on hand for the storage. “

When planning a lesson in which this problem should be solved by the students I would like to ask them to cope with the following request: *Calculate the economic optimal order for quantities of 1.5-V-batteries for the company!* This type of formulation might help the students to open the problem at least in two different mathematical ways, which will be outlined later on and which might help the students to get a closer insight in mathematical modeling. It could and should be discussed in mathematics instruction to which extent the given data represent already an idealization of a real economic situation and what could be optimal for the company in this situation. Students should come to the insight, that it is most reasonable for a company to bring the total cost of delivery and storage to a minimum. Therefore, to determine the economic optimal order quantity one has to calculate that order quantity, which makes the total cost of delivery and storage – including the sum of the expense for purchase and storage – minimal. If the concept of “average inventory on hand” is not known to the students, they have to discuss this concept first before they have the chance to determine it.

3. Possible ways of solution

Step 1: Analyzing, investigating possibilities

First of all questions should be answered, in particularly questions like the following ones: Which parameters are constant in this economic situation? Which are variable? Where has the company a scope of action?

Whenever one tries to model a situation it is very important to analyze the real situation. In order to solve this problem it is not only important to make a distinction between constant and

[2] Schöwe/Knapp/Borgmann, 1996

variable quantities, but it is also crucial to take into account that all variable quantities are depending on the number of orders per period. Therefore the company can influence the total cost of delivery and storage if it varies the number of orders.

Constant quantities are:

- the price of the batteries: 1 € / piece,
- the price of an order: 100 € / order,
- The need for 48 000 pieces/period.

Variable quantities are:

- the number of orders per period; the *following quantities depend on this number* :
- the expenses for purchase,
- the overage inventory,
- the expense for storage.

Step 2: Working forward (to draw first conclusions from given information)

By analyzing some concrete cases, trends could be realized and first hypotheses could be generated.

number of orders	**1**	**2**	**3**	**4**
expense for purchase	100 €	200 €	300 €	400 €
overage inventory on hand	$\frac{48000}{2} = 24000$	$\frac{48000}{2 \cdot 2} = 12000$	$\frac{48000}{2 \cdot 3} = 8000$	$\frac{48000}{2 \cdot 4} = 6000$
expense for storage	24000 ·0,15 € = 3600 €	12000 ·0,15 € = 1800 €	8000 ·0,15 € = 1200 €	6000 ·0,15 € = 900 €
total cost of delivery and storage	3700 €	2000 €	1500 €	1300 €

If you make 1, 2, 3 or 4 orders per period you can observe that the total cost of delivery and storage have a decreasing trend. Therefore you can hypothesize that the total cost of delivery and storage might decrease until infinity if you let the number of orders increase. In addition, you might raise the question whether there is any minimum of the total cost of delivery and storage. Furthermore, there is a clear *functional* dependence between the numbers of orders (only natural numbers are possible) and the total cost of delivery and storage.

However, manipulating discrete quantities is close to reality so students can feel not only comfortable with the problem but also motivated by the problem. From a mathematical point of view the problem suggests the formation of the concept of sequence and also operations (concrete: addition) with sequences. Because of this context it is obvious that it is possible to look on sequences as special functions. Furthermore, the question whether the total cost of delivery and storage have a minimum can initiate the formation of the concepts of boundedness and monotony.

Step 3: Formation of a discrete model by means of mathematics

The total cost of delivery and storage can be determined as the sum of expense for purchase and for storage. Let n be the number of orders, then the expense for purchase is as follows: $P(n) = 100n$ and the expense for storage: $S(n) = \frac{48000}{2n} \cdot 0.15$. So we get for the total cost of delivery and storage:

$$T(n) = P(n) + S(n) = 100n + \frac{48000}{2n} \cdot 0.15 = 100n + \frac{3600}{n}$$

Step 4: Translating the goal (detection of that number of orders which guarantee the minimum of total cost of delivery and storage) into the language of the mathematics

So we are looking for the minimal value of the sequence $T(n) = 100n + \frac{3600}{n}$ if it exists (this holds of course always in praxis, because there can exist only a finite number of values).

Step 5: Solving the mathematical problem

How to test now an infinite but discrete set of numbers for a possible minimum? To pursue this question the following three approaches seem to be - among others - appropriate:

Version a) Estimating by using of the arithmetic and geometric mean

$$\frac{100n + \frac{3600}{n}}{2} \geq \sqrt{100n \cdot \frac{3600}{n}} = 600$$

leads to:

$$100n + \frac{3600}{n} \geq 1200,$$

therefore the sequence $T(n)$ has a lower boundary, the minimal value is reached, if $100n = \frac{3600}{n}$, which has n= 6 as a consequence.

Version b) Checking the monotony of the sequence by calculating differences

$$T(n+1)-T(n)=\left(100\cdot(n+1)+\frac{3600}{n+1}\right)-\left(100n+\frac{3600}{n}\right)=$$

$$=100+\frac{3600}{n+1}-\frac{3600}{n}=100-\frac{3600}{n(n+1)}\begin{cases}<0, & \text{if } 1\le n\le 5\\ >0, & \text{if } 6\le n\end{cases}$$

The sequence $T(n)$ is strictly decreasing if $1\le n\le 5$ and it is strictly increasing if $6\le n$. The minimal value is reached if $n=6$.

Version c) Checking the monotony of the sequence by calculating quotients

$$\frac{T(n)}{T(n+1)}=\frac{100n+\frac{3600}{n}}{100\cdot(n+1)+\frac{3600}{n+1}}=\frac{100n^2+3600}{100\cdot(n+1)^2+3600}\cdot\frac{n+1}{n}=$$

$$=\frac{100n^3+100n^2+3600n+3600}{100n^3+200n^2+3700n}\begin{cases}>1, & \text{if } 1\le n\le 5\\ <1, & \text{if } 6\le n\end{cases}$$

The sequence $T(n)$ is strictly decreasing if $1\le n\le 5$ and it is strictly increasing if $6\le n$.

The minimal value is reached if $n=6$.

Step 6: Reporting the results (Translating the results of the mathematical problem back into the language of economy)

The total cost of delivery and storage is minimal, if the company places 6 orders per period. In this case the company has a total cost of delivery and storage of $T(6)=100\cdot 6+\frac{3600}{6}=1200$ € and therefore the optimal order quantity is 8000 pieces.

Another way for getting a solution:

After step 2 it is also possible to model the functional relation between the number of the orders per period and the incurred expenses by a continuous function. This approach cannot only motivate the formation of the concept of operations (concrete: addition) between continuous functions but it can also motivate - by the question about the existence of minimal total cost of delivery and storage - the concept of boundedness, monotony and extreme values by looking on functions from a practical point of view. Furthermore, students can be motivated in the mathematics lesson to build a relation between the monotony and the existence of extreme values; therefore this problem can also serve as a starting point to introduce the concept of derivate. By this very simple example the drawing of graphs could be much better clarified. I sketch here a possible solution:

Step 1 and 2 are the same as step 1 and 2 in the solution above.

Step 3: Formation of a continuous model by means of mathematics

The total cost of delivery and storage can be determined as the sum of expense for purchase and storage. Let x $(x \geq 1)$ be the number of orders, then we get the following expense for purchase: $P(x) = 100x$ and for storage: $S(x) = \frac{48000}{2x} \cdot 0.15$. So we get for the total cost of delivery and storage:

$$T(x) = P(x) + S(x) = 100x + \frac{48000}{2x} \cdot 0.15 = 100x + \frac{3600}{x}, \quad x \geq 1.$$

Step 4: Translating the goal (see above) into the language of mathematics

We are looking for the minimal value of the function T: x $\rightarrow$ $T(x) = 100x + \frac{3600}{x}$ $x \geq 1$, if it exists.

Step 5: Solving the mathematical problem

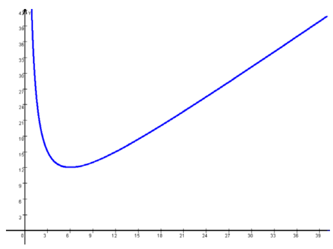
Version a) Checking for extreme values by derivation

$T'(x) = 100 - \frac{3600}{x^2} = 0 \Rightarrow x = 6$ is the only stationary point of the function (because $x \geq 1$).

$T''(x) = \frac{3600}{x^3}$ $\quad T''(6) = \frac{3600}{6^3} = \frac{100}{6} > 0 \Rightarrow$ the function $T(x)$ has a minimum in $x = 6$.

Version b) Checking the monotony of the function

x	$1 \leq x < 6$	$x = 6$	$6 < x$
$f'(x)$	-	0	+
$f(x)$	↘	minimum	↗



Step 6: Reporting the results (Translating the results of the mathematical problem back into the language of economy).

The function of the total cost of delivery and storage has a minimum in $x = 6$. Because the number of orders can only be a natural number, so $x = 6$ is the minimum of the function with an appropriate value of the minimal total cost of delivery and storage.

The total cost of delivery and storage is minimal, if the company places 6 orders per period. In this case the company has a total cost of delivery and storage of

$$T(6) = 100 \cdot 6 + \frac{3600}{6} = 1200 \text{ €}$$

and therefore the economic optimal order quantity is 8000 pieces.

4. Discussion

The mentioned problem treated above does not only offer the possibility for real problem solving but it also allows the formation of some mathematical concepts (sequence; monotony, bounded sequence and operations between sequences; monotony, bounded functions and operations between functions) and it can also motivate some basic concepts of calculus. Furthermore, through contrasting a discrete with a continuous mathematical model the nature of mathematics as a tool for modeling can be discussed. It is possible that a variety of solutions might cause the question whether the model developed here might be transferred to another context. This could induce the debate about possible generalization of solution approaches, and also about their possible limits. Investigating some other versions of the problem some further connections might be discovered between the problem parameters and its possible solutions. For example, substituting the given values for: the price of the batteries (1 €), the price of an order (100 €), the need for batteries per period (48000 pieces) or the amount of expense for storage (15 %) by other concrete values and finally, coming to parameters. For example, if we change in the basic problem only the need of the company for 48000 batteries into 51000 batteries, the minimum in the way of the solution with continuous function will be no natural number.

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Mathematics in Comprehensive School: The Development of Pupil Thinking, Verbalization and Problem-Solving Skills

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Abstract

The aim of this study is to examine teaching and learning methods aimed at developing the mathematical thinking and problem-solving skills of pupils. One of the purposes of this study is that pupils will learn to explain their thinking and problem-solving processes as well as the mathematical solution. The method used in this study was created in 2000 by a group of Mathematics teachers and further developed and implemented with a group of ninth graders (N=15) in Helsinki during the academic year of 2007-2008. During this time period, the method was also implemented with three additional groups (N=9, N=24, N=19). In each group, the pupils solved five different problems. At the end of the academic year the pupils were asked in a questionnaire what they had learned about problem solving. The math lessons, the pupil solutions and the questionnaire were documented and analyzed. It seems as this method was new and quite challenging, it took some time for some pupils to learn, but most of them thought that they had become better problem-solvers.

Introduction

I began by asking the pupils how they would define what a problem is. The answers were mostly that a problem is a very difficult mathematical task that you cannot even solve. Then I asked them to think about if they have problems in their everyday life. Most of them answered: “Yes, some may have problems with their parents, with their health and some people may have mental problems too. But if I want to buy something, maybe some clothes, or I have to take a bus but I did not have any money, this would be a very big problem.” After that we discussed when and where we need problem-solving skills and what kind information and skills we need to solve our problems, including Mathematics.

Mathematical thinking and problem solving have been studied for decades all over the world. Polya (1969) said: My personal opinion is that the main point in mathematics teaching is to develop the tactics of problem solving. He also said; not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems. By Polya teaching is not a science; it is an art.

Theoretical background

About mathematical thinking and problem-solving: Mason asks what mathematical is. It is a dynamic process which expands our understanding by enabling us to increase the complexity of ideas we can handle. What improves mathematical thinking? Practice with reflection. What supports mathematical thinking? An atmosphere of questioning, reflecting and challenging material, along with ample space and time. Where does mathematical thinking lead? To a deeper understanding of yourself, to a more coherent view of what you know, to a more effective investigation of what you want to know, to a more critical assessment of what you hear and see. Sustaining mathematical thinking requires more than just getting answers to questions, no matter how elegant the solution or how difficult the question (Mason, J.& Burton, L. & Stacey, K. 1985, 158-159). According to Polya (1969) the general aim of mathematics teaching is to develop in each pupil as much as possible the good mental habits of tackling any kind of problem.

Why do you need problem-solving skills? According to Polya (1973), a teacher of Mathematics has a great opportunity. If she or he challenges the curiosity of his or her pupils by providing them with problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, she or he may give them a taste for, and some means of, independent thinking. His famous problem-solving process contains four parts: understanding the problem, devising a plan, carrying out the plan and looking back using heuristic strategies in teaching.

Also according to Pehkonen and Zimmermann the teachers can offer open learning environments, within them pupils can deal with real, existing problems and be active and learn in natural settings in which learning happens by investigating and looking for solutions of problems (Pehkonen & Ahtee, 2005; Vaulamo & Pehkonen 1999; Pehkonen & Zimmermann 1990).

According to Schoenfeld (1992) teaching problem solving is hard for the teacher: mathematically; the teachers must perceive the implications of the pupils' different approaches: pedagogically; the teachers must decide when to intervene, and what suggestions will help pupils while leaving the solution essentially in their hands, and carry this through for each pupil: and personally; the teacher will often be in the position of not knowing; to work well without knowing all the answers requires experience, confidence, and self-awareness. Also by

Schoenfeld (1992) pupils can also ask three questions while solving the task: First; what exactly are you doing? Can you describe it precisely? Second; why are you doing it? How does it fit into the solution? And third; how does it help you? When the teacher starts to ask these questions early in term, by the end of the term this behaviour has became habitual. Problem solving is also constituted by resources, strategies, control, beliefs (Schoenfeld, 1985)

There are closed and open problems. In this research both closed and open problems were used. Problems are defined as open if their goal is not provided, e.g. such as in investigations, problem posing, real-life situations, projects, problem fields, problems without a question and problem variations. When teachers are using open tasks in mathematics teaching, pupils have an opportunity to become a creative mathematician. The crucial component here is the pupil's own creative power (Pehkonen & Zimmermann 1990; Pehkonen & Ahtee, 2005).

About verbalizing and studying in the classroom: According to Lee (2006) the atmosphere in the classroom has to be conducive to open-ended thinking that encourages mathematical discussion as a process so that pupils do the thinking and articulate their ideas. Pupils need to support one another to develop a common understanding. Both pupils and teachers must not be afraid to also explore "incorrect" answers. The teacher emphasizes that they are looking for ideas and explanations (Lee 2006).

The teacher should give pupils enough time and mulling ground, when providing mathematical problems for discussion. Pupils should also have enough time to think about a problem and to discuss possible solutions together. They can start with the problem in class and continue solving the problem and the thinking process at home, and write their ideas down (Pehkonen & Ahtee 2005). For a teacher, providing mathematical problems in class is also an opportunity to learn from one's pupils. According Pehkonen and Ahtee, communication between a teacher and his/her pupils will improve, when the teacher shows that she/ he is trying to understand the thinking processes of pupils. Pehkonen and Ahtee have studied the way teachers listen to their pupils: not listening, selective listening, evaluative listening, interpretative listening and open listening. The teachers' aim is to make a change possible in the pupils' thinking (Pehkonen & Ahtee 2005).

Learning to use language to express mathematical ideas may be similar to learning to speak a foreign language for many pupils. Pupils need to learn to express their mathematical ideas, but they need guidance. Unless pupils comprehend the way language is used in mathematics, they may think that they do not understand a certain concept since they are unable to express the idea in words. Pupils have to learn specific vocabulary and means of phrasing and expression that are specifically mathematical and which make it possible to explain mathematical ideas (Lee 2006).

Research question

How do the pupils solve the tasks when verbalizing in thinking and problem-solving process as well as the mathematical solution is required, and how does this method contribute to teaching in the class?

Implementation of the study

The aim of this action research was to develop thinking, verbalizing and creative problem-solving skills of pupils, to teach the problem-solving process and to study how this method contributed to teaching in class.

I teach mathematics in Helsinki at the comprehensive school and during the 2008-2009 academic year I examined the method, used in this study, in three mathematics groups that I teach. Each of these groups solved five different problems. There was a group of eight pupils in the 7th grade (13-14 year olds), a group A of nineteen pupils in the 9thgrade group A (15-16 year old) and group B of twenty-four pupils in the 9th grade (15-16 years old). The groups were heterogeneous; pupils with different skills, and personal problems, and pupils with other native languages than Finnish. The tasks were chosen just to be tested in each of these groups. The pupils solved problems both during class and as homework assignments. For instance the 9th grade A pupils solved following tasks: task 1; hunting for numbers, task 2; the age of Diofantos; task 3; the house problem, task 4; the rumour, task 5; the snail problem. The lessons were filed documented and analyzed. The pupil processes of problem-solving were evaluated, documented, and partly analyzed later. At the end of the term the pupils were also asked, by means of questionnaire, what they had learned about problem solving. The answers were filed and analyzed.

The documentation of the lessons was divided into four parts: first; when the pupils received the task, second; when the pupils solved the problem (mostly at home) and returned the task, third; when the teacher evaluated the tasks and fourth; when the pupils got their feedback. The pupils discussed their problem-solving ideas with each other and with the teacher in class.

The pupils received instructions "How to do it" (Polya 1973; Schoenfeld. 19992) including several questions to help with the problem-solving process. They were asked to explain their thinking process 1. Understanding the problem; How do you understand the problem? (easily/ it was very difficult/ I had to read many times/ I had to ask my mother..) What is unknown? What are the data? 2. Devising a plan; How are you going to solve the problem? What are you doing? 3. Carrying out the plan; How did you solve the problem? Why are you doing it? Carrying out your plan of for the solution and checking each step. Writing all the steps and methods you have used, what kind of problems did you have, how your solution is going? 4. Looking back. Can you check the result? Can you derive the result differently? Can you use the result, or the method, for some other problem? These instructions were often discussed in the classroom.

The teacher evaluated the tasks as follows: Excellent (10): You have taken into account all parts of the problem and demonstrated creative thinking. Very good (9)**:** You have taken into account all parts of the problem and understood the mathematics behind the problem. Your answer is clear and organized. Good (8-7): You've understood the problem. Your explanation is insufficient or contains a slight error. Good effort (6): You haven't completely understood the problem. You haven't explained your answer. Little effort (5): You've tried a little, e.g. written the problem down. During the lessons the grading criteria were discussed and most pupils understood how they get excellent grades or why they get good grades. (e.g Nykänen & al, 2000). Also these grading criteria were discussed during lessons and especially when the pupils got their feedback.

Results

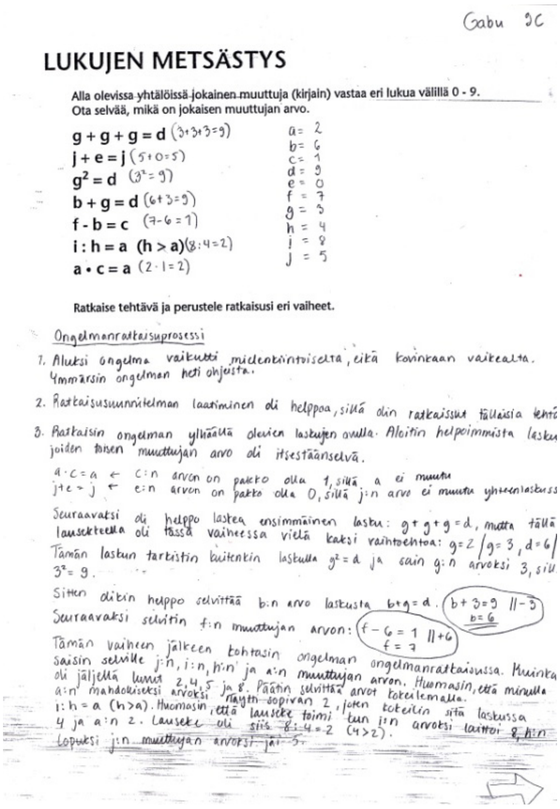
1. Giving the task for the pupils

Anthon: "Hey teacher…why don't the other groups have to solve any problems at home?" Another comment of Anthon:" I already solved that problem" and then he shouted the answer, which was correct and then he continued" ...but I only have numbers in my head… " And the teacher tried to encourage and motivate Anthon:" ...Try to write those numbers on the paper…" And at the end of the spring term Anthony asked the teacher:"…well do you now want to hear how I solved that problem? I hate to write..."

In most lessons the teacher motivated the pupils by saying:" Today you will learn something new and interesting…you may remember our problems….” Also the problem-solving process, as the evaluation criterions were discussed. Most of the pupils like to solve problems but for some of them, especially writing is difficult. They would just like to tell or guess the answer.

The task “Hunting for numbers” was for instance the first task for the pupils in the 9th grade A. The first problem was chosen because it was not too difficult and the pupils could easier practice the explanation. The pupils discussed, argued and solved the task mostly in small groups. Some of them had forgotten mathematical rules and others helped them. The pupils liked the problem and everyone could solve it. As homework, they got to write the problem-solving process with all their ideas, explanations and thoughts.

The task “Hunting for numbers”: In the clues below, each variable represents a different digit 0-9. Determine the value of each variable.
g + g + g = d, j + e = j, gxg = d, b + g = d, f – b = c, i / h = a (h > a), a x c = a
(New standards 1997, US).



Gabu 9C

LUKUJEN METSÄSTYS

Alla olevissa yhtälöissä jokainen muuttuja (kirjain) vastaa eri lukua välillä 0 - 9.
Ota selvää, mikä on jokaisen muuttujan arvo.

g + g + g = d (3+3+3=9)	a = 2
j + e = j (5+0=5)	b = 6
g^2 = d (3^2 = 9)	c = 1
b + g = d (6+3=9)	d = 9
f - b = c (7-6=1)	e = 0
i : h = a (h > a) (8:4=2)	f = 7
a • c = a (2·1=2)	g = 3
	h = 4
	i = 8
	j = 5

Ratkaise tehtävä ja perustele ratkaisusi eri vaiheet.

Ongelmanratkaisuprosessi

1. Aluksi ongelma vaikutti mielenkiintoiselta, eikä kovinkaan vaikealta. Ymmärsin ongelman heti ohjeista.
2. Ratkaisusuunnitelman laatiminen oli helppoa, sillä olin ratkaissut tällaisia tehtä…
3. Ratkaisin ongelman ylhäällä olevien laskujen avulla. Aloitin helpoimmista lasku… joiden toisen muuttujan arvo oli itsestäänselvä.

a·c=a ← c:n arvon on pakko olla 1, sillä a ei muutu
j+e=j ← e:n arvon on pakko olla 0, sillä j:n arvo ei muutu yhteenlaskuss…

Seuraavaksi oli helppo laskea ensimmäinen lasku: g+g+g=d, mutta tällä lausekkeella oli tässä vaiheessa vielä kaksi vaihtoehtoa: g=2 / g=3, d=6 / … Tämän laskun tarkistin kuitenkin laskulla g^2=d ja sain g:n arvoksi 3, sill… 3^2=9.

Sitten olikin helppo selvittää b:n arvo laskusta b+g=d. (b+3=9 ||-3 b=6)
Seuraavaksi selvitin f:n muuttujan arvon: (f-6=1 ||+6 f=7)
Tämän vaiheen jälkeen kohtasin ongelman ongelmanratkaisussa. Kuinka saisin selville j:n, i:n, h:n ja a:n muuttujan arvon. Huomasin, että minulla oli jäljellä luvut 2, 4, 5 ja 8. Päätin selvittää arvot kokeilemalla. a:n mahdolliseksi arvoksi näytti sopivan 2, joten kokeilin sitä laskussa i:h=a (h>a). Huomasin, että lauseke toimi kun i:n arvoksi laittoi 8, h:n 4 ja a:n 2. Lauseke oli siis 8:4=2 (4>2).
Lopuksi j:n muuttujan arvoksi jäi 5.

Figure 1. Gabu’s task

Gabu wrote: 1. Understanding the problem; At first the problem seemed to be very interesting and quite easy. I understood the problem right away. 2. Devising a plan; It was easy to make the developing plan because I had solved similar kinds of problems. 3. Carrying out the plan; I solved the problem with the clues from above. I started from the easiest task a x c=a, and c=1 because a is the same. Then j+ e = j, so e had to be 0, because j is the same. And so Gabu continues her story… 4. Looking back; At the end I checked my result and counted the task and everything seemed to be all right. She also told that the task was very easy, but nice and she worked alone. It took five minutes to solve the task and write the solution.

2. The pupils give their tasks for the teacher

Paula: “If I solve that problem can I give it to you tomorrow? “ Teacher: “Yes of course.” Kim: “I will give it to you on Monday OK?” Laura: “I cannot solve it even if I try very hard. “ Helen:” I have solved it, but I want to write it more exactly.” Harry:” I have solved it with Len and now I do not remember it anymore.” Teacher: “So you do not remember how you solved this problem.” Harry: “Yes and I had already made a devising plan.” Larry: “Then you just write it.” Teacher: “Sally, have you solved this problem?” Sally: “It was so terribly difficult.” Mike: “I have not got any paper at all.” And the other pupils discussed and compared their solutions while giving them to the teacher.

When the pupils returned their homework to the teacher it was usually quite a chaos in the classroom. There were always pupils who had not solved any problems or they had forgotten their work at home or they had lost their papers. There were also very positive comments. For example Tim said happily: “The task was so easy. I solved it in ten minutes”. Many of the pupils needed positive support and some asked questions. Sally: “For me it took the whole day to solve it”. Ann: “I did not understand. What does it mean to make a devising plan? Should I do that?” Gabu returned her task in time.

Table 1 Pupils’ returned tasks, math homework

	7 %	9A%	9B%
Task 1	100	31	61
Task 2	75	56	70
Task 3	75	31	52
Task 4	88	44	52
Task 5	100	100	100
altogether	87,6	52,4	67
homework	90%	80%	79%

The 7th grade pupils solved 87, 6% of their tasks - almost the same amount as their homework: 90%. It was a small group and it was easy for the teacher monitor the pupils. The 9th grade A pupils solved 52, 4% of their tasks which was less than their homework: 80%. In task 1: “hunting for numbers” the pupils solved the problem during the lesson, but many of them did not return the work to their teacher. Task 3: “the house problem” and task 4:” the rumour” was too difficult for many. The 9th grade B pupils solved 67% of their tasks, which is also less than their normal homework: 79%. Task five motivated all the pupils to do their best and to turn in their work.

3. Evaluating the tasks

Effective feedback also by Lee (2006) helps pupils to know how to move forward with their learning and allows them to spend time thinking through and talking about the task they are undertaking. Written feedback always takes more time. In a classroom where the focus is on using language to learn mathematics, pupils will receive feedback constantly. As they talk through their own ideas with one another they will hear their own thought processes. This is a valuable way to check that their ideas make sense.

The teacher who evaluated the tasks also wrote short feedback. The evaluation was very important for the pupils and they compared their grades. There were pupils who tried very hard to solve problems and they often explained their solution attempts, but they had problems understanding the task.

Table 2 Grades of the returned tasks, math grades, grades in mother tongue

	7	9A	9B
Task1	8,6	9,2	8,7
Task 2	8,3	9,1	8,5
Task 3	8,3	7,8	7,1
Task 4	8,0	8,8	7,7
Task 5	7,9	7,5	7,5
grades/returned am	8,2	8,5	7,7
grades/all am	7,2	4,4	5,3
Math grades am	9	8,2	7,7
Mother tongue am	8,2	7,8	7,6

Gabu got “very good” of her task. She had taken into account all parts of the problem and her answer was clear and organized. In all groups, the average level of the returned tasks was ”good”. The pupils had understood the task but their explanation was insufficient or contained a slight error. The 7th grade pupils solved the problems, but they could not quite explain their ideas. The 9th grade A pupils succeeded quite well in their explanations and solved their problems. If we take a look of all returned tasks the mean value is very low with 9th graders although the grades in Mathematics and Mother tongue were good.

4. Giving feedback

Helen: “That was so difficult and a terrible task and it took so long time for me to solve it and to write.” Teacher: “What was difficult and terrible in the task?” Helen: “Well, it was just so terrible.” Kim: “Because you can solve the task up until a certain point and then you just stuck and you cannot go further.” Teacher: “Did you notice that you had made a mistake?” Kim: “No, you can just solve it up to a certain point and no further.” Teacher: “So you finished your clues?” Kim and some other pupils: “Yes that happened to me.” Sally: “And I just didn’t understand where to start. It was so difficult.” Teacher: “All right. Mary, Mike and Nina you solved this task. Would you like to give some tips? And maybe we can look at the task again and try to solve it. Would you like to do that?”

These are some of the answers to the 9th grade A pupils for why they couldn’t solve the task3.

The pupils were eager to have the teacher return their work and they were very happy when they had succeeded. The pupils openly discussed and compared their ideas and problem-solving abilities. Those pupils who had difficulties understanding or who had made some mistakes were sitting quietly. The teacher gave feedback and encouraged them

Discussion

Prior to this study, many of the pupils had had only some experience with problem-solving as a process. But the pupils solved the tasks fairly well, even though many of them needed a lot of support. Also, many of the pupils did not solve all five tasks. It seems that some of the tasks were too difficult or different. It is very important that teachers know their pupils and their skills in mathematics. It is also very important to give ample time for discussion. During the lessons, the pupils wanted share their ideas and different ways of solving the task. Pupils who had problems in learning mathematics often asked for help from other pupils and in that way they were able to solve the task and they also received positive feedback. On the other hand, some pupils found problem solving very difficult because they needed to remember mathematical concepts from their earlier studies. Many pupils also found it difficult to explain their ideas and solutions in writing and, thus, their descriptions were problematic. For the teacher it is essential to notice those pupils and to motivate them. Some of the pupils said that they did not want to think very much; they just wanted to know the right solution. But there were also plenty of pupils who wouldn’t give up and wanted to solve the problem even it took time. Many pupils were also very busy with their other homework and hobbies. One particular problem is how to encourage and motivate those pupils who have experienced difficulties in understanding mathematics or those pupils who have lost hope in their ability to learn and understand mathematics.

By Carpenter and Lehrer (1999) in classrooms that emphasize understanding, learning tasks are viewed as problems to be solved, not exercises to be completed using narrowly defined procedures. Learning is viewed as problem-solving rather than practice and drill. These classrooms are discourse communities in which pupils discuss alternative strategies or different ways of viewing important mathematical ideas. Pupils expect that the teacher and their peers will want explanation as to why their conjectures and conclusions make sense and why a procedure they have used is valid for a given problem. Mathematics becomes a language for thought rather than merely a collection of ways to get answers.

The 7th grade pupils felt that the best task was inventing the dice game. During the lessons they played their games with the teacher. One of the 9th grade pupils said that he had started to think about problems in a different way. One pupil claimed that she had learned to express her thinking process more clearly and she has used problem-solving skills in her everyday life, as well as in her other studies. Some pupils also said that writing their solutions helped them to think, remember and understand, as well as to notice their mistakes.

By Pimm (1995) part of learning to talk like a mathematician is to be able to use language both to control and conjure personal mathematical images, as well as to convey them to others. Also pupils need to learn how to use mathematical language to create, control and express their mathematical meanings as well as to interpret the mathematical language of others.

For the teacher, creating new learning environments with problem-oriented teaching is challenging. The discussions of pupils in the corridors and the inspired atmosphere in the classroom, as well as student mathematical solutions all point to the need to continue to teach using the problem-solving approaches described here. (c.f. Turunen, 2009).

In the future it is important to create tasks corresponding the needs and skills of pupils. Also it seems that the grading criteria used in this study needs to be clarified.

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Comparisons between Cypriot and English undergraduate primary education students' beliefs about mathematical problem solving

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The research reported in this paper draws on semi-structured interviews conducted with first year undergraduate teacher education students, in the first weeks of their course at one university in Cyprus and one in England. The interviews, focused on students' conceptions of mathematical problems and problem solving yielded substantial, culturally located, variation in students' responses highlighting continuing inconsistencies in the operationalisation of this key concept around the world. Some implications for teacher education and further research in the problem solving field are discussed.

Introduction

Over the last two or three decades, teachers' beliefs have been the subject of extensive research, based on the assumption that what teachers believe is a significant determinant of what gets taught, how it gets taught and what gets learned in the classroom (Middleton 1999; Chapman 2002; Wilson and Cooney 2002). According to Aguirre and Speer (2000: 327), *being able to identify and describe the mechanisms underlying the influence of beliefs on instructional interactions would deepen and enrich our understanding of the teaching process*. Older and recent studies highlight the importance of examining, analysing and changing teachers' beliefs in order to implement successfully mathematics curricula reforms (Ernest 1989; Handal and Herrington 2003). Yet, without a challenge to their underlying beliefs, so the arguments go, teacher may exploit new resources or modify practice inappropriately (Cohen 1990; Handal and Herrington 2003).

Beliefs have been defined *as conceptions, personal ideologies, world views and values that shape practice and orient knowledge* (Aguirre and Speer 2000: 328). Moreover, recent beliefs-related research, continuing to draw on Ernest's (1989) triadic model, has focused primarily on how teachers think about the nature of mathematics, its teaching and its learning. Additionally, drawing upon Bandura's early work in the field (Bandura 1977), research has highlighted the role of teacher self efficacy in general (Wolters and Daugherty 2007; Charalambous et al 2008) and mathematics teaching self-efficacy in particular (Xenofontos

2009a). Thus, we argue, Ernest's original model is appropriately augmented by such additional dimensions.

Research has shown that the relation between beliefs and instructional practices is complex and cannot be described simply in terms of cause-and-effect. Indeed, although a number of studies have highlighted substantial disparities between espoused and enacted beliefs (Thompson 1984; Cohen 1990; Beswick 2005; Raymond 1997), other studies have indicated that both beliefs and actions are contingent on the changing nature of the classroom context (Schoenfeld 2000; Skott 2001). Thus, acknowledging this problem, we have examined the mathematics-related beliefs of beginning undergraduate primary teacher education students to determine the extent they reflect similar, culturally embedded, perspectives to their peers. Indeed, in this respect, results from comparative studies of serving teachers' mathematics related beliefs (Andrews and Hatch, 2000; Santagata 2004; Correa et al. 2008) and practices (Givvin et al. 2005; Andrews 2007a) indicate that culture plays a key determinant role in both their formation and manifestation.

Curricula reforms, teachers' beliefs and problem solving

Much recent beliefs-related research, particularly in respect of mathematics education, has been located within the context of reform (Saxe et al. 1999; Cady et al. 2007). Reform classrooms are characterised by an emphasis on problem solving and connections within both mathematics itself and within the real world (Peterson et al. 1989; Cady et al. 2007). As such, notions of reform have framed a number of recent studies examining the mathematics instruction-related problem-solving beliefs of pre-service teachers in, for example, Flanders (Verschaffel et al. 1997) and the US (Timmerman 2004). In many such studies, pre-service teachers' cultural location remained, as a significant influencing variable, essentially unacknowledged. Moreover, a collective definition of *problem-solving-oriented instruction* is frequently assumed and, it is our contention, although this paper is not the place for a lengthy elaboration, that this assumption has little basis. For instance, as a consequence of the role of the National Council of Teachers of Mathematics (NCTM) in the framing of reform curricula in the US, much problem solving research has been undertaken in that country. The results of these studies have influenced curriculum development in many countries, and variation in definition and implementation can be seen, for example, in a 2007 special edition of ZDM on problem solving around the world that includes articles highlighting the role of problem solving in the curricula of Israel, France, Italy, the UK, the Netherlands, Portugal, Germany, Hungary, China, Australia, Singapore, Japan, Brazil, Mexico, and the US respectively. In other words, despite US influence in the field, both problem solving as an activity and problem solving research continue to mean different things in different countries, to the extent that problem solving, according to where and by whom the term is used, can mean a goal, a process, a basic skill, a mode of inquiry, a form of mathematical thinking, and a teaching approach (Chapman 1997). Significantly, in this respect, Xenofontos (2009b) has identified at least four fields in which cultural differences have been essentially neglected in problem solving research, particularly in respect of the meanings ascribed to the term by participants. These are research trends on problem solving in different countries; the curricular importance and justification of problem solving; pupils' beliefs and competence in problem solving; and teachers' beliefs, competence and practices in problem solving.

In this paper we examine prospective primary teachers' beliefs about mathematical problems and problem solving in Cyprus and England. The cultural similarities and differences of their beliefs are seen as part of the fourth perspective above, which we call *the teachers' perspective.* While some single-national studies of teachers' problem solving beliefs and practices have been undertaken in, for example, Australia (Anderson et al. 2004) and Cyprus (Xenofontos and Andrews 2008), few cross-national studies have been undertaken in this area, although, from the perspective of serving teachers' beliefs about mathematics, Andrews (2007b) concludes that English teachers tended to view mathematics as applicable number and the means by which learners are prepared for a world beyond school, while Hungarian teachers perceived mathematics as *problem solving* and independent of a world beyond school. Such findings confirm, it seems to us, that the *teachers' perspective* is a neglected dimension in comparative studies of problem solving.

Mathematical problems and problem solving

There is much agreement amongst scholars as to the nature of a problem. One key characteristic is that a problem lies with the person seeking the solution and not the problem itself. As Schoenfeld (1985: 74) notes, *being a 'problem' is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person*. That is, a problem for one person may not be for another (Borasi, 1986; Nesher et al. 2003). Such insights, and our summary of the work of these scholars, are helpful in framing our study, not least because they allude to three key criteria for defining the relationship between problem and problem solver. Firstly, problem solvers must have encountered a block and see no immediate and obvious way forward, secondly, they must actively explore a variety of plausible approaches to the problem and thirdly, they must accept that the search for a solution necessitates an engagement with the problem. This perspective frames the study we report below.

Method

As stated above, in this paper we report on the problem and problem solving beliefs of prospective primary teachers from Cyprus and England. Participants were in the first weeks of an undergraduate teacher preparation programme at a one reputable, as measured by systemic measures of teacher education accountability, university in each country. Data were collected by means of semi-structured interviews at the beginning of the academic year 2008-2009. The Cypriot cohort comprised thirteen students (twelve female, one male), while the English comprised fourteen (thirteen female, one male). At the time of the interviews participants had received no problem solving-related university instruction. Therefore, they were seen as products of the school rather than university systems of their countries. Analyses were focused on students' *meaning* (see Kvale and Brinkmann 2009) and drew on both theory-driven and data-driven approaches (Boyatzis 1998; Kvale and Brinkmann 2009). In this paper, due to space limitations, we report on three of the ten themes identified by the analyses. These are students' perspectives on the nature of mathematical problems, mathematical problem solving and the characteristics of effective problem solvers. The results are presented alphabetically by nationality, Cyprus then England.

Cypriot students' perspectives on mathematical problem

Several perspectives permeated the Cypriot students' responses. Eleven indicated that mathematical problems, usually embedded in text, should be clearly presented with adequate information and data so that solvers can easily attempt a solution. Panayiota's comments were typical of others. She said that a mathematical problem comprises *mathematics related sentences, which include information, data and a desired outcome. We have to think about the data, to process them and get the answer*. Eight students commented on the significance of difficulty in defining a problem. For some as reflected in Sofia's comment, a problem by definition *has* *difficulty and unknown factors within it.* Others indicated that notions of difficulty lay, essentially, with the problem solver and not of itself, the problem. Such a perspective could be seen in Demetra's comment that *problems and their difficulty are connected to certain age groups*, while Haroula added that *the criterion is school, if it is primary, gymnasium or lyceum. In primary school, problems are very easy, in gymnasium they are more complicated, and in lyceum you can find the hardest*. All thirteen students implied that mathematical problems are contextualised within a real-world framework. Christina's comment was not atypical. She said,

> *at the first grade, problems were like "I have two apples, my grandmother gave me two more, how many do I have now?" Later on, at the sixth grade, problems were more complex, let's say something about how many square metres of a wall surface could someone paint with so many litres of paint. In gymnasium, they might be something like "how much it cost to paint a surface", which had to do with area and volume. In lyceum, they were more or less the same*.

Cypriot students' perspectives on mathematical problem solving (MPS)

Eleven students indicated, either directly or indirectly, that MPS is a process. Of the four who used the word process explicitly, Pantelis' comments were typical. He said that *mathematical problem solving is a process, the process towards what we are asked to find. It is the process during which you use the given data in order to find the answer to a problem*. Of the others, Demetra's comment was typical. She said that *mathematical problem solving is the use of the data in order to find what you are asked to*. A recurrent theme in these students' responses was the need to read the problem repeatedly. Panayiota's comment was typical. She said,

> *You have to read the problem two-three times, underline some key points, because you know, sometimes problems have unnecessary things in them, you have to find what is important, then start processing all these in your mind, read two-three more times, write down your data and what you want to find, do a shape if it's needed and then do the algorithms.*

Cypriot students' perspectives on what makes a good problem solver

Eight students suggested, along the lines of Martha's comments that *good problem solvers have the skills for organising the given information quickly. They tidy up the data, the ques-*

tions. Those... who are not good don't structure their work. Sofia presented a typical response in respect of distinguishing the expert from the novice. She said that *there is a big difference. Someone who is a good solver, as soon as he sees the problem, he has a clear picture in his mind about what has to be done, directly. Someone... who is less good... will have difficulties in finding which way to follow for solving the problem*. Several students added that problem solving requires concentration, as seen in Demetra's comments that *solvers who concentrate when they encounter a mathematical problem... perceive what has to be done quickly and manage to resolve it*.

Eight students also suggested that, with practice, problem solving competence can be acquired. For example, Angeliki commented that a novice *could spend more time on practice... to develop his mathematical thinking, learn about different types of problems, and develop a clearer idea around mathematical problems*. However, the remaining five students indicated that being a good problem solver was natural. As Panayiota noted, it all *depends on the individual, biologically,. ...I think... some people are born with it; it's their talent... either you have it or not.*

English students' perspectives on mathematical problems

In respect of their conception of mathematical problems, the English students presented a range of perspectives although common to ten was an explicit invocation of number operations, as seen in Victoria's comment, that a problem was *anything, from adding, dividing, subtracting, timesing, or arrange them and then put together*. However, the context in which they described their perspectives varied. For four students problems were essentially mathematical in nature, as seen in Daniel's slightly recollection of his school experiences. He said

> *Pythagoras' theorem, if you have the length of the hypotenuse and you need to work out,.. I can't remember how it was. Like we've got sin, cosine and tangent and you need to work out the other two. Like you've got one of the angles and you need to work out another angle or length. That's one which sticks in my head.*

Two students indicated that mathematical problems were related to the real world and everyday life, as in Laura's comment that they had to do *with money, we did how, if apples cost 10p each, how much money do I need to buy six apples? Which is 60. Six times 10p.* However, the majority of the group, seven students, implied that problems could be construed as either mathematical or real world. Melanie's comment reflected those of others. She said that a *mathematical problem* could mean *lots of things. It can be a standard two plus two on a piece of paper or how much money I need to go for shopping. (...) Something that uses numbers to come up with 'a' answer, or a series of answers*.

English students' perspectives on mathematical problem solving (MPS)

Eleven students indicated that m*athematical problem solving* is a structured process during which solvers apply prior knowledge in a structured step-by-step approach. Julia's response, typical of others. Suggested that MPS was *just basing what skills you know on trying to solve a problem in maths, so just applying the knowledge to structure it and work step by step to work out a problem,* while for Laura it meant having to *take it step by step and apply things you already know.* For some this step-by-step approach meant breaking down a problem into smaller pieces-tasks, working on each piece separately, and finally putting all the pieces together. Rachel, reflecting comments of others, said that *I break it down..., and then do a bit a time and then at the end I put them all together. I do that with most mathematical problems*. All students saw MPS as drawing on prior knowledge. Interestingly, Melanie, was the only student who used the word *process* explicitly. She commented on the

> *the processes that you use to solve a problem. So, the way you think, the way you work out a problem, whether you need a resource to do it or whether you do it in your head. What steps you take to come to your conclusion. You need to understand the problem, what you've been asked to find out, cause lots of problems are in a context where you have to pull out the information you need, you need to understand the processes, you need to follow out the procedure and then you need to understand your answer.*

English students' perspectives on good problem solvers

Thirteen students suggested that expert problem solvers can see through problems and apply the necessary mathematical knowledge and strategies quickly and efficiently. For example, Laura indicated that *an expert knows how to answer straight away, whereas someone who is not so good does think a long time about it and mainly have one option, whereas an expert might have lots of different ways to think about it.* Expert solvers, she added, *already have the knowledge to work out what you need to do to solve the problems, whereas, otherwise they* *have to think what steps you need to take to get there.* Daniel summarised the nature of expertise thus, *I would say an expert is kind, they already have the formulas in their head so they can just work it out mentally.*

The same thirteen students indicated that a prerequisite of expertise was practice. Lucy's comments were typical. She said

> *I think is just practice than anything else... if you're learning a language, you have to practice it, don't you? To learn it. So, like if they have different, if they have theories, like an example of how to solve a problem, and if they practice and do it over and over again, like just memorise it and you know how to use that theory then I think that would get better.*

Discussion

Space limits the extent of our discussion, although a number of important outcomes have emerged from the analysis that merit comment. Despite some within country differences, the major variation lay between countries. Firstly, The Cypriot students tended to see problem as located in text, a perspective shared by many researchers and cultures (see, for example, both Flemish and Hungarian national curricula). Also, there was a clear sense that problems were, essentially, characteristics of the problem solver. Such perspectives went largely unseen in the English responses where context – whether real world or mathematical world - rather than process seemed more important. Also, the Cypriot students tended to talk in general, almost abstract, terms while the English in particularities. For example, in defining a problem many Cypriot students focused on the generic characteristics of a problem while the English tended to offer examples of problem types from which properties could be inferred by the reader.

Similar issues emerged with respect to the nature of problem solving. Both groups of students attended, in some way, to process. The interesting difference lay in the sense that Cypriot students tended to view the process holistically – read, understand, collect data, analyse data and so on – while the English saw the process as one of simplification or reduction of the task to a series of small steps. Such differences are unlikely to be coincidental. Inevitably they will reflect the teaching these students received prior to going to university. Indeed, the English perspectives of applicable number find resonance with an earlier study of English and Hungarian teachers' beliefs about the nature of classroom mathematics (Andrews 2007b) and the process of simplification with an earlier study of English and French curricular traditions (Jennings and Dunne 1996) Importantly, from the perspective of future teacher education programmes, these students are not mathematics majors but prospective generalist primary teachers. They are people who, one day, will teach young children mathematics. If problem solving is a key element of that country's curriculum, then English universities clearly need to understand the beliefs their undergraduates bring to their studies. The problems would appear less severe, at least as far as beliefs are concerned, for the Cypriot authorities.

In terms of their beliefs about the characteristics of effective problem solvers there was evidence of genuine similarity across the two groups, with substantial proportions comparing the approaches of effective problem solvers – the ability to see straight through the problem to a solution strategy with no apparent difficulty - with those of novice or inefficient problem solvers. Moreover, there was also a consensus that such skills could be acquired through appropriate practice. Lastly, such findings highlight the plea made earlier that those involved in research on problem solving, at all levels, need to acknowledge not only the lack of definitional consensus but also the key role played by culture in the construction of participants' construal of mathematical problem, problem solving and the characteristics of effective problem solvers.

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“Open ended Problem Solving in Mathematics Instruction and some Perspectives on Research Questions” revisited – New Bricks from the Wall?

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Abstract

The paper “Open ended Problem Solving in Mathematics Education and some Perspectives on Research Questions” had been published some twenty years ago. This issue should be revisited and analyzed (by looking back, at the present time and into future) in a broader context especially with along the following key-points: Status of open ended mathematical problems - Status of teaching and learning mathematics - Status of research methodology - Psychological questions related to the single pupil - Mathematical content - Educational questions.

Key words

mathematical problem solving, open ended problems, research in problem solving, change and progress in problem solving

Introduction

In my ZDM-paper (Zimmermann 1991) I presented:

1. Some remarks about the problem solving discussion at that time.
2. A discussion of the question “What are open-ended problems?” including
    a) a preliminary definition of open-ended and closed problems,
    b) the method of opening tasks by variation and augmentation, demonstrated along the example of the multiplication table (10x10),
    c) Characteristics of open-ended problems.
3. Some goals, which might be achieved by using open-ended problems in mathematics instruction.
4. Some reasons for teaching open-ended problem solving. Especially
    a) changes in mathematical belief-systems - including a new picture of school mathematics,
    b) philosophical reasons for teaching problem solving,
    c) deficits of the present school-system.
5. Some open research question concerning problem-solving.

Why to revisit the paper from 1991?

The main reason is to look for possible change and progress in the specific domain.

There are important events, which took place during the last nearly twenty years which have influence on such processes of change and which will be taken into account in the following analysis:

- The change of the world by the disappearance of the "iron curtain", with the German reunification as a major consequence. Of course, specific "pictures of men" and different teaching philosophies (belief-systems) become more prominent.
- Results of TIMSS and PISA have - at least in Germany - a considerable impact on the situation at schools.
- The emergence of the internet and new technical tools for learning.

In the present analysis I want to revisit the following issues drawn from that paper:

1. Status of open ended mathematical problems,
2. Status of teaching and learning mathematics,
3. Status of research methodology,
4. Psychological questions related to the single pupil,
5. Mathematical content,
6. Educational questions.

Each of these themes should be analyzed according to the following aspects:

- Situation or deficits until about 1990 as to the paper of Zimmermann 1991,
- Change until today,
- Looking ahead.

I concentrate especially on the educational situation in Germany - to Western Germany before 1990 and to the reunified Germany after 1990. Some reference is made, too, to the educational situation in Finland (which I know quite well from my own experience) and Hungary (about which I hope to learn more in the future).

1. Status of open ended problems in mathematics education

Situation until 1990:

Since at least the 70ties of the last century there had been an increasing call not only for a focus on problem solving, but also on more open ended problem solving in mathematics education (cf. Brown/Walter 1983, Zimmermann 1977, Zimmermann 1983, 1986, Pehkonen 2003). Nevertheless the degree of realization and implementation of these ideas into school-praxis was rather poor.

Change until today:

At the turn of the decades from the eighties to the nineties of the last century at least four major developments concerning open-ended problem-solving became more prominent:

- There had been a tremendous increase of the number of *concrete examples and material for open ended problems* as well as opening and variation of problems (including school-books; cf. e.g. Blum/Drüke-Noe/Hartung/Köller 2006, Cukrowicz/Zimmermann 2000).
- More *comprehensive books for student-teachers, teachers and researchers* were offered (cf. e. g. Becker/Shimada 1997), including special issues as variations of problems (cf. Schupp 2002) and constructing own problems by teachers, student teachers or pupils (Büchter/Leuders 2005).
- More emphasis on *modelling and the combination of (open-ended) problem solving and modelling* can be observed (cf. Greefrath MU 2010).
- There is an increasing amount of *empirical* studies including open ended problems (cf. eg. Pehkonen/Zimmermann 1990).

These trends had been reinforced at least in Germany by the outcomes of the well-known TIMSS- and PISA-studies.

Especially the "revival" of empirical studies and the turn to the "output" of educational efforts had been triggered by these studies.

Looking ahead:

Taking into account the need for more precise outcome of educational efforts to teach and learn open-ended problem solving, a more precise conceptualization of the research-field of open-ended problems might be necessary and useful.

For reasons of better communication it might be helpful, to go to the past first and to remind aspects of openness with respect to the givens, the question and the possible solution of a problem. A problem might be open or closed (intermediate-states are possible, of course) at least with respect to the following aspects (cf. Dörner 1976, this system includes some additional components to that one of Büchter/Leuders 2005, p. 93).

A very comprehensive classification-system could be found in Zimmermann 1977, p. 20 - 54.

2. Status of teaching and learning of mathematics

Deficits until 1990:

1. As to the *picture of mathematics and mathematics instruction* there is a long history of claims concerning the rigidity about the understanding of mathematics instruction at German schools. Since the time of M. Ohm from the 19th century until now mathematics is taken very often as a "complete consequent system".
2. The praxis concerning *learning and thinking* was centred very often - at least until that time - to rote memory-learning and performing simple routine-tasks. There was a major

lack of thinking in relationships.

3. The *methods of measuring (assessing) achievement* determined very often the content goals of teaching and not vice-versa. So there were less open-ended problems and process-oriented evaluation at least in Western Germany at that time.

Change until today:

1. There seems to be some local change of the *picture of mathematics and mathematics instruction* in the direction of that one which has been presented in Zimmermann 1991, p. 40. In Germany e.g. the SINUS-project[1] - triggered by TIMSS - helped teachers to think more about every-day-applications (also now in Hungary), and open-ended problem solving (in Finland already for a longer time).
2. It seems that a change of the philosophy about *learning and thinking* took place by a re-discovery of old principles as education for pupil's independent thinking and active discovery-learning[2] and are now often labelled by "self-regulation". Especially in Eastern Germany and Hungary there seems to be now increasing emphasis on pupils' independent thinking processes.
3. If we have a closer look an *methods of measuring (assessing) achievements* one can assume that at least in Germany (tests to compare achievements in nearly all Federal States) another re-discovery of old principles took place as a consequence of outcomes of international studies. Very often there seems to be still (or again) the following opinion: What can be measured, is important and not: What is important, should be measured (as far as possible). So it is quite typical, that in the new educational standards ("Bildungsstandards") in Germany the goal "to be creative in mathematics (on an appropriate level)" - yet an important part in the list of the educational goals in mathematics instruction of Winter 1975 - does not appear. There might occur to some extent the same problems as in the sixties and seventies of last millennium related to goal-directed instruction (Mager 1994), only with another vocabulary (standards, measuring competencies). This danger is admitted[3], but it is not clear how it can be avoided or at least diminished.

Looking ahead

1. *Picture of mathematics and mathematics instruction*: It is obvious that the picture of mathematics and its teaching of any teacher and researcher is shaped by the foregoing experience. So it is in my case. My experience in different parts of Western and Eastern

[1] cf. SINUS 2007.
[2] cf. e.g. Comenius 1891, p. 132, 176.
[3] Blum/Drüke-Noe/Hartung/Köller 2006, p. 18.

Germany as well as experience abroad (esp. Finland, but also in the US and Hungary) helped me, to learn not only new views of mathematics, but helped to learn something about the shaping process of such views, too. So I come to the following suggestion: A picture of mathematics with many facets is very helpful. To determine the own one, one has to have the possibility to learn about others so that one can contrast and become aware of the specific own view. Well known books as those from Davis/Hersh 1981, Courant/Robins 1996 or Lakatos 1976 are quite useful to initiate thinking about once view on mathematics. But this is no substitution for own experience with teaching and learning of mathematics in very different situations and at very different places.

2. Conceptions of *learning and thinking* should stress even more the independence (freedom![4]) of pupils - connected of course with the learning about once own responsibilities and duties. So teachers and researchers might incorporate and think much more about "outdoor-learning", minimal instruction and the role of school as a "pit-stop"-place for getting some "learning-fuel" on the own tour of learning mathematics (cf. Haapasalo/Zimmermann/Eronen 2007).
3. One should try to better harmonize important educational goals and methods/instruments to assess the "learning-output". More appropriate instruments are to be developed to fit better to the educational goals and needs than vice versa.

3. Status of research methodology

Deficits until 1990:

1. Especially in psychological research there had been frequently a *restriction on isolated aspects of teaching and learning* – e. g. on affective, social or cognitive aspects (very often focussed only on the pupil or on the teacher) mainly at the expense of other important aspects.
2. Furthermore there had been a severe *lack of implementation-strategies* of new ideas into educational praxis. One of the standard methods of adoption of new concepts by teachers had been: Adopt of the language of the new wave and describe your old instruction by this new vocabulary, so you can say: We do it already![5]
3. There was a major gap between the goals of mathematics-educators and teachers. This seemed to be due to a lack of cooperation between researchers and teachers, too. E. g. more action research might help to change the situation.

[4] „Das Wesen der Mathematik liegt in ihrer Freiheit" (The essence of mathematics is freedom) Georg Cantor

[5] cf. Thompson 1992, p. 143, Baumert/Lehmann 1997.

Change until today:

1. Today there seems to be *less restriction on isolated aspects* and more awareness of the complexity of teaching and learning problem solving (cf. e. g. Fritzlar 2003) and real-classroom-settings (cf. e.g. Leppäaho 2007).
2. One can observe also a *decrease of the lack of implementation(-strategies)* concerning teaching and learning problem solving. In Germany the SINUS-project - initiated by the outcomes of TIMSS - should be quoted once again, in Finland - even more comprehensive - the LUMA-project - which started already before PISA - help to bring ideas about problem solving into the classroom.
3. Since several years there are some praxis-oriented projects (in Germany, Finland and the EU) which can be taken as indicators that the gap between mathematics researchers and teachers seem to become smaller (e. g. Pehkonen/Zimmermann 1990, Haapasalo/Zimmermann/Eronen 2007, SINUS 2007, Leppäaho 2007, LEMA).

Looking ahead

1. Research should take even more into account the *complexity of learning and teaching of mathematical problem solving*. International studies which compare problem solving achievements in different countries have to consider much more the specific societal boundary conditions as the general esteem of teaching and learning, the image of teachers and the "hidden contracts" and systems of values. As to the experience of me there are (in this respect) e.g. considerable differences between Germany and Finland which had not been analyzed by TIMSS or PISA at all (see below!).

2. As to the experience of the author one of the most *effective implementation strategies* (not only) for problem solving strategies is: visit schools, give your own lessons let other teacher observe it, observe lessons of students, talk about just observed lessons and combine it directly with (general) principles of problem solving. So I have done for one year at a German "Hauptschule" (compulsory school for lower achievers). The experience was very encouraging (see below).

3. So it might help much to bridge the gap between researchers and teachers if more researchers would like to work and "live" with teachers at school for a longer time and - on the contrary - if (at least some) teachers have the possibility to make (part work) research at the university. At the University of Jena we are just starting such approach in relation with a special exercise-term (half year full time teaching) for student-teachers at school.

4. Psychological questions related to the single pupil

Deficits until 1990:

1. For a long time there had been a *restriction on the mathematical learning processes of the individual pupil*, additionally very often in *laboratory studies*. Models of mathematical learning processes concentrated along the cognitive-science approach on cognitive aspects mainly (e. g. computer-simulation models of human problem solving). Such models were often developed on the expense of meaning and understanding-processes in problem solving. Affective aspects could not be modelled as well.
2. *Brain research* at that time produced mainly some hypotheses drawn from results from hemisphere-analyses which could hardly be related to mathematical problem solving.
3. *Belief-research* was at the very beginning at that time. In some studies evidence was found that the belief about the nature of mathematics had impact on the way of teaching (problem solving).
4. *Research on individual differences* had its main focus on different achievement and abilities. There was a lack of investigations of thinking styles of pupils and teaching styles of teachers.
5. Most problems from psychological studies at that time had a *lack of relevance and authenticity* with respect to every day-situation.
6. Papert 1993 published with his well-known book "Mindstorms" the first overarching conception in which a technological dominated learning-environment should facilitate to do, to "speak" and to learn mathematics. There were strong connections to AI and Piaget's learning theory; additionally motivational and genetic aspects were stressed as well as a productive treatment of mistakes in mathematical problem solving.

Change until today:

1. There seems to be a clear trend to studies which have their *focus on the whole classroom-setting* and the respective social interactions and learning processes (cf. e.g. Leppäaho 2007).
2. New *techniques as MRT* and improvements of old ones (as EEG) opened quite new doors to analyze the relation between brain- and problem-solving processes. This brought new possibilities to test and to refine old learning theories and the corresponding results have some impact on the teaching and learning of mathematics (including problem solving), too (cf. e.g. Raichle 1994; Seidel 2001, Spitzer 2002).
3. There had been *many studies on mathematical beliefs*, which had a closer look on their impact on mathematics instruction, especially on (open-ended-) problem-solving, too. (cf.

e. g. Zimmermann 1991, 1996, 1997, 1998, Leder/Pehkonen/Törner 2002, the work of the MAVI-group; Stipek et al 2001, Haapasalo/Zimmermann/Eronen 2007).

4. *Individual differences* become more and more important also in learning and teaching of problem-solving. This is due not only to the fact, that more different types of problem-solvers were discovered (e. g. Rehlich 1995; Zimmermann 1992) and that pupils with migration-background need also specific educational respect and treatment.
5. As a consequence of PISA there is a trend (revival!) to more *real-life problems, authenticity and modelling* (cf. e. g. the work of the ISTRON-group). But it is also obvious, that such goals, which are declared by some representatives of the PISA-community, are not always reached. This might become unravelled by a careful analysis of published "real-life"-problems (cf. e.g. Kießwetter 2002).
6. There was a tremendous boost concerning new IT-guided learning-environments during the last twenty years. This is not only caused by the emergence of the internet, but also by the development of many new hardware devices (Inspire; CLASSPAD etc.) and new software tools (CAS, DGS, spreadsheet, …). So there is empirical evidence, that - e. g. - DGS might stimulate open-ended problem-solving and investigations (reinforcing the generation of hypotheses), but their use might also lead to a decrease of the need to prove something (cf. e.g. Hölzl 1994). In latest studies modern learning theories (conceptual/procedural knowledge; minimal instruction methods) and modern technological devices and software are combined to stimulate change of belief-systems of students, including their self-esteem, as well as their mathematical problem-solving competencies (cf. e. g. Haapasalo/Zimmermann/Eronen 2007).

Looking ahead

It can be assumed that all trends just described will be reinforced in the future. The domain of individual differences will be augmented by the need to care more for pupils with migration background. Therefore bilingual mathematics instruction (and problem solving) will become even more important in the future in all European countries (cf. e. g. Szücs 2008).

5. Status of mathematical content

Deficits until 1990:

1. The *mathematical content* of tasks or problems from studies with orientation on the cognitive science approach was *not very rich* or stressed more formal aspects until that time.

2. *The mathematical content* of tasks or problems from cognitive-psychologists' studies was mainly selected to *fit to the respective evaluation instruments* or at least to one and only one "competence".
3. At that time there had been *too much stress on procedural, too less on conceptual knowledge* so that understanding was not one of the major concerns of such studies.
4. *History of mathematical heuristics and problem solving* was a neglected area until that time. It might be possible that such studies might help to give additional tools (besides learning theories) to a better understanding of students' problem solving processes, furthermore some consequences might be derived from such studies concerning explicit or implicit heuristics teaching in problem-solving instruction.

Change until today:

1. In cognitive studies as well as in (international) studies of the outcome of problem-solving instruction there had been some *trend towards richer mathematical content* (cf. e. g. Dörner 1989, Blum/Drüke-Noe/Hartung/Köller 2006). But there is - e. g. in the Netherlands - still a lot of criticism concerning "realistic mathematics instruction" (the core of the PISA-Philosophy). Already some years ago I could experience some severe criticism to statements of de Lange about the "realistic approach" by a teacher from the Netherlands. He claimed that there had been a decrease of mathematical content in the classroom-praxis in the Netherlands. Now the situation seems to improve (cf. Kaenders 2009).
2. The *strong impact of evaluation-instruments* (to check the "output") on the mathematical content of the respective problems as a consequence of PISA-like tests which are administered periodically - unfortunately - might reduce again mathematical content (cf. the philosophy of the IQB in Berlin).
3. Permanent testing might again reinforce the procedural gain of knowledge and reduce understanding.
4. The results of a comprehensive *analysis of the history of mathematical heuristics and problem solving* (cf. Zimmermann 1991a) had been integrated into the framework of a new school-book-series (cf. Cukrowicz/Zimmermann 2000, Zimmermann 2003) which has as a main guide line problem-solving (including open-ended problem-solving). Furthermore the detection of eight fundamental activities which proved to be fruitful in the production of mathematics along some 5000 years, helped to construct a framework for determination and evaluation of the development of pupils' belief-systems during a course on self-regulated problem-solving with incorporation of modern technologies (cf. Haapasalo/Zimmermann/Eronen 2007).

Looking ahead

1. *Mathematical content:* It seems to be important, that the mathematical content meet standards of good mathematics, relevance and authenticity for pupils and appropriateness for the stage of development of the respective students in a good equilibrium. A permanent discussion is necessary where experts from at least all three domains (including mathematicians, teachers, practitioners and educators) are involved.
2. For me there seems to be some danger that the *evaluation instruments* (due to the focus on "output-orientation") *dominate* (determine to much) *the input*. Once again: The general educational objectives and the content should be discussed and determined first and the evaluation-instruments have to be developed according to this decision.
3. There should be conducted empirical studies which test, if the use of background knowledge for teachers about *history of mathematical problem-solving* might help them to understand better their pupils and help them to improve the problem-solving abilities of their pupils.

6. Educational questions

Deficits until 1990:

1. How to teach heuristics and problem-solving? Until that time there were different results concerning the effect of teaching and learning of heuristics in problem-solving.
2. What about intercultural differences concerning problem solving? There were hardly any concrete results before 1990.
3. How to cope with differences in problem solving achievements? Corresponding to results from studies from well-known researchers as Weinert et al. I recommended less striving for equalization and more acceptances of differences concerning problem solving achievements.
4. How to combine open instruction and open ended problem solving? This question was hardly analyzed.

Change until today:

1. How to teach heuristics and problem solving? There have been several studies during the last years which seem to repeat well-known designs from the seventies. E. g., in the educational standards ("Bildungsstandards Mathematik" 2003) is written that the pupil should "select" appropriate strategies. There is no hint, that the pupil might also re-construct them himself, which might help to deeper root these methods in the mathematical problem-solving inventory of a pupil. There seem to be a "roll-back"-movement to explicit

teaching of problem-solving strategies. Depending on the quality of problems as well as of the instruments of evaluation, of course, there is again the possibility to prove this approach as successful. There remain still the experiences with case studies including lower achievers (from a German "Hauptschule") who proved to be quite successful (and motivating) problem-solvers (determining minimal spanning trees, tower-of-Hanoi solutions and generalizations) without any explicit training of problem-solving strategies (creating their strategies themselves).

2. What about intercultural differences concerning problem solving? Especially TIMSS and PISA constituted a tremendous increase in the amount of international studies. But there remain several deficits concerning a careful analysis of the cultural and social background of the respective countries (see above).
3. How to cope with differences in problem solving achievements? There seem to be no really new suggestions and methods for this domain.
4. How to combine open instruction and open ended problem solving? There are many methodological suggestions to combine open instruction and open-ended problem-solving (cf. Büchter/Leuders 2005).

Looking ahead

There remain at least six major issues for the future:

1. Good mathematics instruction to improve problem solving abilities and understanding
2. Good problems
3. Good sequencing of problems and good orchestration of problem sets (to meet better individual difference and come to a better understanding)
4. Good methods for implementation
5. Good teachers for problem-solving
6. Good culture of discussions of all involved participants

Good mathematics teachers are of course a must for better teaching results, too!

Therefore math-teachers have to have at least that experience and competence in problem solving they expect their pupils to get. Special entrance-examination for university training (as in Finland) might be considered in Germany, too.

As to the last issue, it can be assumed that there will be a much broader discussion (across countries) about methods of teaching heuristics and problem solving.

There will be much more encounters of intercultural differences because of freedom of choosing the place of living e. g. in the European Union.

One has to take into account much more the fact that there are tremendous differences in the status of school and schooling, mathematics (mathematicians) and teachers in different countries (e. g. between Finland, Germany and Hungary). The differences between systems of values (of teachers, pupils, parents, math-educators) should become much more aware, analyzed and by discussion be propagated.

So the open problem of good mathematics teaching (problem solving including open ended problem-solving) could enrich and educate all engaged participants of this discourse.

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